Some congruence theorems for closed hypersurfaces in Riemann spaces
(The continuation of Part III)

Dedicated to the memory of Professor Dr. Heinz Hopf

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Introduction. This is the continuation of the previous paper ([1]) given by H. Hopf and the present author. In [1], considering an \((m+1)\)-dimensional orientable Riemann space \(S^{m+1}\) with constant curvature of class \(C^* (\nu \geq 3)\) which admits a one-parameter group \(G\) of isometric transformations, we proved the following

Theorem. Let \(W^m\) and \(\bar{W}^m\) be two orientable closed hypersurfaces in \(S^{m+1}\) which do not contain a piece of a hypersurface covered by the orbits of the transformations and \(p \bar{p}\) be the corresponding points of these hypersurfaces along an orbit, and \(H_r(p)\) and \(\overline{H}_r(p)\), \(r=1, \cdots, m\) be the \(r\)-th mean curvatures of these hypersurfaces at \(p\) and \(\bar{p}\) respectively. Assume that in case \(r \geq 2\), the second fundamental form of \(W^m(t) = (1-t)W^m + t\bar{W}^m, 0 \leq t \leq 1\), is positive definite. If the relation \(H_r(p) = \overline{H}_r(p)\) holds for each point \(p \in W^m\), then \(W^m\) and \(\bar{W}^m\) are congruent mod \(G\).

In the present paper, we shall cancel the assumption that the transformations are isometric, in fact, under a group \(G\) of essentially arbitrary transformations it is the purpose of the present paper to generalize the above theorem. Especially, in case of \(r=m\), that is, the general theorem relating to the Gauss curvature was already proved in the previous paper [2].

§ 1. A certain integral form for two closed hypersurfaces. We suppose an \((m+1)\)-dimensional orientable Riemann space \(S^{m+1}\) with constant curvature of class \(C^* (\nu \geq 3)\) which admits an infinitesimal transformation

\[ \xi^t = x^t + \xi^t(x) \delta \tau \]

(where \(x^t\) are local coordinates in \(S^{m+1}\) and \(\xi^t\) are the components of a contravariant vector \(\xi\)). We assume that orbits of the transformations generated by \(\xi\) cover \(S^{m+1}\) simply and that \(\xi\) is everywhere continuous and \(\neq 0\). Let us choose a coordinate system such that the orbits of transfor-

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1) Numbers in brackets refer to the references at the end of the paper.
mations are new $x^i$-coordinate curves, that is, a coordinate system in which the vector $\xi^i$ has components $\xi^i = \delta_{j}^{i}$, where the symbol $\delta_{j}^{i}$ denotes Kronecker's delta; then (1.1) becomes as follows

\begin{equation}
\hat{x}^i = x^i + \delta_{1}^{i}\partial \tau
\end{equation}

and $S^{m+1}$ admits a one-parameter continuous group $G$ of transformations which are 1–1-mappings of $S^{m+1}$ onto itself and are given by the expression

\begin{equation}
\hat{x}^i = x^i + \delta_{1}^{i}\tau
\end{equation}
in the new special coordinate system.

Now we consider two orientable closed hypersurfaces $W^m$ and $\overline{W}^m$ of class $C^\nu$ imbedded in $S^{m+1}$ which are given as follows

\begin{equation}
\begin{cases}
W^m : & x^i = x^i(u^\alpha) \quad i=1, \ldots, m+1 \\
\overline{W}^m : & \bar{x}^i = x^i(u^\alpha) + \delta_{1}^{i}\tau(u^\alpha)
\end{cases}
\end{equation}

where $u^\alpha$ are local coordinates of $W^m$ and $\tau$ is a continuous function attached to each point of the hypersurface $W^m$. We shall henceforth confine ourselves to Latin indices running from 1 to $m+1$ and Greek indices from 1 to $m$.

Then we can take the family of the hypersurfaces

\begin{equation}
W^m(t) = (1-t)W^m + t\overline{W}^m \quad 0 \leq t \leq 1,
\end{equation}
genrated by $W^m$ and $\overline{W}^m$ whose points correspond along the orbits of the transformations, where $W^m$ and $\overline{W}^m$ mean $W^m(0)$ and $W^m(1)$ respectively. Thus according to (1.3), $W^m(t)$ is given by the expression

\begin{equation}
W^m(t) : \quad x^i(u^\alpha, t) = (1-t)x^i(u^\alpha) + tx^i(u^\alpha) \quad 0 \leq t \leq 1,
\end{equation}
and (1.4) may be rewritten as follows

\begin{equation}
W^m(t) : \quad x^i(u^\alpha, t) = x^i(u^\alpha) + \delta_{1}^{i}t\tau(u^\alpha) \quad 0 \leq t \leq 1.
\end{equation}
The relation between $\overline{W}^m$ and $W^m(t)$ becomes as follows

\begin{equation}
\bar{x}^i(u^\alpha) = x^i(u^\alpha) + \delta_{1}^{i}(1-t)\tau(u^\alpha).
\end{equation}

If we take the hypersurface $W^m(t_0)$ defined by a fixed value $t_0$ in $0 \leq t \leq 1$, then we have the transformation $T_{(1-t_0)\tau(p_0)}$ in $G$ attached to the point on $W^m(t_0)$ corresponding to $p_0 \in W^m$, given by

\begin{equation}
T_{(1-t_0)\tau(p_0)} : \quad \hat{x}^i = x^i + \delta_{1}^{i}(1-t_0)\tau(u^\alpha_0),
\end{equation}

\begin{equation}
(1-t_0)\tau(u^\alpha_0) = \text{constant}.
\end{equation}
Thus we get the additional hypersurfacen
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\[ \overline{W}^m_{p_0}(t_0) \overset{\text{def.}}{=} T_{(1-t_0)r(p_0)} \cdot W^m(t_0) \]

which passes through the corresponding point \( \overline{p}_0 \) on \( \overline{W}^m \), and is given by

\[ \overline{W}^m_{p_0}(t_0):\quad \overline{x}_{p_0}^s(u^a, t_0) = x^s(u^a, t_0) + \delta_1^s(1-t_0)\tau(u_0^a), \]

\[ (1-t_0)\tau(u_0^a) = \text{const}. \]

Therefore we have the additional hypersurfaces

\[ \overline{W}^m_{p}(t) = T_{(1-t)\tau(p)} W^m(t) \quad 0 \leq t \leq 1 \]

for all hypersurfaces in the family, which pass through the corresponding point \( \overline{p}_0 \in \overline{W}^m \). Thus we can consider \( \overline{W}^m_{p}(t) = T_{(1-t)r(\overline{p})} W^m(t) \) for each point \( p \in W^m \), which pass through the corresponding point \( \overline{p} \in \overline{W}^m \), and the normal unit vector \( \overline{n}^s(t) \) of \( \overline{W}^m_{p}(t) \) at \( \overline{p} \).

Let us give henceforth the derivative with respect to \( t \) by the dash. We shall calculate \( \tilde{n}^s(t) \). Then \( \bar{g}_{ij} \) being the metric tensor of \( S^{m+1} \) at \( \bar{p} \) and differentiating the following relations with respect to \( t \),

\[ \bar{g}_{ij}\overline{n}^i_p(t) \frac{\partial \overline{x}^j_p(u, t)}{\partial u^a} = 0, \quad \bar{g}_{ij}\overline{n}^i_p(t)\overline{n}^j_p(t) = 1, \quad 0 \leq t \leq 1 \]

since \( \bar{g}_{ij} \) is independent with respect to \( t \), we have

(1.6) \[ \bar{g}_{ij}\overline{n}^i_p(t) \frac{\partial \overline{x}^j_p(u, t)}{\partial u^a} + \bar{g}_{ij}\overline{n}^i_p(t) \frac{d}{dt} \left( \frac{\partial \overline{x}^j_p(u, t)}{\partial u^a} \right) = 0, \]

(1.7) \[ \bar{g}_{ij}\overline{n}^i_p(t)\overline{n}^j_p(t) = 0. \]

From (1.6), (1.7) and

\[ \frac{d}{dt} \left( \frac{\partial \overline{x}^s_p(u, t)}{\partial u^a} \right) = \frac{d}{dt} \left( \frac{\partial x^s(u, t)}{\partial u^a} \right) = \delta_1^s \tau_a, \]

we obtain

(1.8) \[ \tilde{n}^s_p(t) = -\bar{g}^{as}_p(t)\tau_a \delta_1^r \overline{n}_{pr}^l(t) \frac{\partial \overline{x}^l_p(u, t)}{\partial u^s}, \]

where \( \bar{g}^{as}_p(t) \) is the contravariant metric tensor of \( \overline{W}^m_{p}(t) \) and \( \tau_a \) means \( \partial \tau/\partial u^a \).

Throughout this paper repeated lower case Latin indices call for summation 1 to \( m+1 \) and repeated lower case Greek indices for summation 1 to \( m \); but \( p \) is not a summation index. And also for the covariant differential of \( \tilde{n}^s_p(t) \) along \( \overline{W}^m_{p}(t) \) at \( \overline{p} \), we get

\[ \delta \tilde{n}^s_p(t) = d\overline{n}^s_p(t) + \bar{T}^s_{jk} \tilde{n}^j_p(t) \overline{x}^k_p du^r, \]

where \( \bar{T}^s_{jk} \) is the christoffel symbol with respect to the metric tensor of
Calculating $(\delta\tilde{n}^{i}_{p}(t))^\prime$, we have

$$(\delta\tilde{n}^{i}_{p}(t))^\prime = (d\tilde{n}^{i}_{p}(t))^\prime + \overline{\Gamma}_{f1}^{i}\tilde{n}_{p}^{f}(t)\tilde{x}_{p\gamma}^{l}du^{r}$$

because of $\overline{\Gamma}_{ fk}^{i}$ is independent with respect to $t$. Consequently we get the following relation between $\delta\tilde{n}^{i}_{p}$ and $(\delta\tilde{n}^{i}_{p})^\prime$

$$(\delta\tilde{n}^{i}_{p})^\prime = \delta\tilde{n}^{i}_{p} + (d\tilde{n}^{i}_{p})^\prime$$

because of $d\tilde{n}^{i}_{p} = (d\tilde{n}^{i}_{p})^\prime$.

We consider the following differential form of degree $m - 1$ attached to each point on the hypersurface $\overline{W}_{p}^{m}(t)$

$$(\tilde{n}_{p}', \delta\tau, \delta\tilde{n}_{p}, \cdots, \delta\tilde{n}_{p}, d\tilde{x}_{p}, \cdots, d\tilde{x}_{p})$$

(1.10)

where $g$ is the determinant of the metric tensor $g_{ij}$ of $S^{m+1}$, the symbol $( )$ means a determinant of order $m + 1$ whose columns are the components of respective vectors and $\tilde{b}_{\alpha\beta}(t)$ is the second fundamental tensor of $\overline{W}_{p}^{m}(t)$ and $\tilde{b}_{\alpha\beta}^{\gamma}(t)$ denotes $\tilde{b}_{\alpha\beta}(t)\tilde{g}_{p}^{\gamma}(t)$.

Then the exterior differential of the differential form (1.10) becomes as follows

$$d((\tilde{n}_{p}', \delta\tau, \delta\tilde{n}_{p}, \cdots, \delta\tilde{n}_{p}, d\tilde{x}_{p}, \cdots, d\tilde{x}_{p}))$$

(1.11)

$$= ((\tilde{n}_{p}', \delta\tau, \delta\tilde{n}_{p}, \cdots, \delta\tilde{n}_{p}, d\tilde{x}_{p}, \cdots, d\tilde{x}_{p}))$$

$$+ ((\tilde{n}_{p}', \delta\tau, \delta\tilde{n}_{p}, \cdots, \delta\tilde{n}_{p}, d\tilde{x}_{p}, \cdots, d\tilde{x}_{p}))$$

$$+ ((\tilde{n}_{p}', \delta\tau, \delta\tilde{n}_{p}, \cdots, \delta\tilde{n}_{p}, d\tilde{x}_{p}, \cdots, d\tilde{x}_{p}))$$

because since $S^{m+1}$ is a space of constant curvature, we have

$$((\tilde{n}_{p}', \delta\tau, \delta\tilde{n}_{p}, \delta\tilde{n}_{p}, \cdots, \delta\tilde{n}_{p}, d\tilde{x}_{p}, \cdots, d\tilde{x}_{p})) = 0.$$

From that the quantity $\tilde{n}_{p}(t)\tilde{\sqrt{g}}_{p}(t)$ is independent with respect to $t$, where $\tilde{g}_{p}(t)$ is the determinant of $\tilde{g}_{p}(t)$, we see
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\[ (\delta \tau, (\delta \tilde{n}_p', \delta \tilde{n}_p, \ldots, \delta \tilde{n}_p, d\tilde{x}_p, \ldots, d\tilde{x}_p)) \]
\[ = (-1)^r m! \mathcal{H}_{pr} \tilde{n}_p \delta \tau d\tilde{A}_p(t) \]

where \( \mathcal{H}_{pr}(t) \) and \( d\tilde{A}_p(t) \) are the \( r \)-th mean curvature and the area element of \( \tilde{W}_p^m(t) \) respectively, and using (1.8), we have

\[ ((\tilde{n}_p, \delta \tau, \tilde{n}_p, \ldots, \tilde{n}_p, d\tilde{x}_p, \ldots, d\tilde{x}_p)) \]
\[ = (-1)^r \tilde{g}_p^{\alpha \beta}(t) \tau_{\alpha} \tilde{n}_p \delta_{1}^{\iota} \tilde{A}_p(t) \]

on the other hand, from (1.9) we have

\[ ((\delta \tilde{n}_p', \delta \tau, \tilde{n}_p, \ldots, \tilde{n}_p, d\tilde{x}_p, \ldots, d\tilde{x}_p)) \]
\[ = ((\delta \tilde{n}_p, \delta \tau, \tilde{n}_p, \ldots, \tilde{n}_p, d\tilde{x}_p, \ldots, d\tilde{x}_p)) \]
\[ - (\tilde{T}_{\gamma} \tilde{n}_p \delta \tau d\tilde{u}', \delta \tau, \tilde{n}_p, \ldots, \tilde{n}_p, d\tilde{x}_p, \ldots, d\tilde{x}_p). \]

Then putting

\[ (m-1)! \mathcal{C}_{pr}^{(r)} = \varepsilon_{\tilde{\alpha}_{1} \cdots \tilde{\alpha}_{r} \cdots \tilde{\alpha}_{m-1} \tilde{\alpha}_{r+1} \cdots \tilde{\alpha}_{m}} \tilde{b}_{\tilde{\alpha}_{r+1}} \cdots \tilde{b}_{\tilde{\alpha}_{m-1}} \tilde{b}_{\tilde{\alpha}_{r}} \tilde{b}_{\tilde{\alpha}_{r}} \]

and using (1.11), (1.12), (1.13), (1.14) and the relation

\[ \delta(\delta_{1}^{\iota}) = \tilde{T}_{\gamma} \tilde{n}_p \delta \tau d\tilde{u}', \]

we have
\[ d\left( \bar{n}_p', \delta\tau, \delta\bar{n}_p, \cdots, \delta\bar{n}_p, d\bar{x}_p, \cdots, d\bar{x}_p \right) \]
\[ = \frac{(-1)^{r-1}}{r} m! \overline{H}_{pr}' \bar{n}_{pi} \delta_1^i \tau \overline{d}\bar{A}_p(t) \]
\[ + (-1)^{r-1}(m-1)! \overline{C}_{pr(r)}^{\alpha\beta} \tau_{\alpha} \tau_{\beta} (\bar{n}_p(t) \delta_1^p) \overline{d}\bar{A}_p(t) \]
\[ + \left( (\bar{n}_p', \tau \overline{\Gamma}_{f1} \delta_{1}^{i} \tau d\bar{A}_p(t) + (-1)^{r-1}(m-1)! \overline{C}_{pr(r)}^{\alpha\beta} \tau_{\alpha} \tau_{\beta} (\bar{n}_p(t) \delta_1^p) \overline{d}\bar{A}_p(t) \right) \]

Furthermore we shall calculate the following quantity

\[ \left( (\bar{n}_p', \tau \overline{\Gamma}_{f1} \delta_{1}^{i} \tau d\bar{A}_p(t) + (-1)^{r-1}(m-1)! \overline{C}_{pr(r)}^{\alpha\beta} \tau_{\alpha} \tau_{\beta} (\bar{n}_p(t) \delta_1^p) \overline{d}\bar{A}_p(t) \right) \]

(1.16)

For the first term of (1.16), making use of (1.8) and from that \( \overline{C}_{pr(r)}^{\alpha\beta} \) is the symmetric tensor, we have the following

\[ (\bar{n}_p', \tau \overline{\Gamma}_{f1} \delta_{1}^{i} \tau d\bar{A}_p(t) + (-1)^{r-1}(m-1)! \overline{C}_{pr(r)}^{\alpha\beta} \tau_{\alpha} \tau_{\beta} (\bar{n}_p(t) \delta_1^p) \overline{d}\bar{A}_p(t) \]

(1.17)

where \( \overline{\Gamma}_{f1} \) is \( \overline{g}_{li} \overline{\Gamma}_{f1}^{l} \) and the symbol \((\gamma\beta)\) means the symmetric part for indices \( \gamma \) and \( \beta \).

On the other hand, putting the vector \( \delta_i^t \) by the following expression

\[ \delta_i^t = (\bar{n}_p(t) \delta_i^p) \bar{n}_p + \overline{\varphi}_p^\beta(t) \bar{x}_p^\beta \]

for the second term of (1.16), we have

\[ -(\overline{\Gamma}_{f1} \bar{n}_p(t) \tau \delta_{1}^{i} \tau d\bar{A}_p(t) + (-1)^{r-1}(m-1)! \overline{C}_{pr(r)}^{\alpha\beta} \tau_{\alpha} \tau_{\beta} (\bar{n}_p(t) \delta_1^p) \overline{d}\bar{A}_p(t) \]

\[ = -(-1)^{r-1} \left\{ n_p(t) \delta_i^t \left( \overline{\Gamma}_{f1} \bar{n}_p(t) \tau \bar{x}_p, \cdots, \bar{x}_{pa_{r-1}}, \bar{x}_{pa_{r}}, \cdots, \bar{x}_{pa_{m-1}} \right) \right\} \]

\[ + \overline{\varphi}_p^\beta(t) \left( \overline{\Gamma}_{f1} \bar{n}_p(t) \tau \bar{x}_p, \cdots, \bar{x}_{pa_{r-1}}, \bar{x}_{pa_{r}}, \cdots, \bar{x}_{pa_{m-1}} \right) \]

\[ \times \delta_p^a(t) \cdots \delta_p^{a_{r-1}}(t) du^a \cdots du^{a_{r-1}} \cdots du^{a_{m-1}} \]
Let us take the relation (1.14) and
\[ \tilde{\epsilon}_{p\alpha_{1}\cdots\alpha_{m+1}}\tilde{g}_{p}(t)\tilde{x}_{p}^{f}\tilde{g} = (-1)^{m}\tilde{\epsilon}_{\alpha_{1}\cdots\alpha_{m+1}}\tilde{x}_{p}^{f}, \]
where \( \tilde{\epsilon}_{\alpha_{1}\cdots\alpha_{m+1}} \) means the \( \epsilon_{\alpha_{1}\cdots\alpha_{m+1}} \) at \( \tilde{p} \). Then after some calculations, we get
\[ \begin{align*}
- \left( \bar{T}_{f}^{i}(t)\delta_{f}^{i} \tilde{\epsilon}_{\alpha_{1}\cdots\alpha_{m-1}}\tilde{x}_{p}^{f}(t) \tilde{x}_{p}^{\alpha_{1}} \cdots \tilde{x}_{p}^{\alpha_{m-1}} \tilde{n}_{p}^{f} \right) \\
(1.18)
&= (-1)^{r-1}(m-1)! \tau \left\{ \tilde{n}_{pl}(t)\delta_{f}^{i}(\bar{T}_{fj}^{i}\tilde{n}_{p}^{j}(t)\tilde{x}_{p}^{l}(t)) \right. \\
&- \bar{T}_{pl}(t)\tilde{n}_{p}^{l}(t)\tilde{n}_{p}^{i}(t)\tilde{\psi}_{p(l}^{\beta} \bar{\psi}_{p\beta)}(t) \right\} \tilde{C}_{p}^{fi\gamma}(t) d\tilde{A}_{p}(t),
\end{align*} \]
where \( \tilde{\psi}_{p\beta} \) is \( \tilde{g}_{p\beta\alpha}(t)\tilde{\psi}_{p}^{\alpha} \).

Thus from (1.17), (1.18) and \( \bar{T}_{jl} + \bar{T}_{jl} = \left( \frac{\partial g_{ij}}{\partial x^{j}} \right)_{\tilde{p}} \), we have
\[ \begin{align*}
(\tilde{n}_{p}', \tau \bar{T}_{j}^{i}\tilde{x}_{p}^{l} du^{l}, \delta_{f}^{i} \tilde{n}_{p}, \cdots, \delta_{f}^{i} \tilde{n}_{p}, \tilde{d} x_{p}, \cdots, \tilde{d} x_{p}) \\
- \left( \bar{T}_{j}^{i}(t)\delta_{f}^{i} \tilde{n}_{p}^{f}(t) \tilde{n}_{p}^{i}(t) \tilde{x}_{p}(t) \right) \\
(1.19)
&= (-1)^{r-1}(m-1)! \tau \left\{ \tilde{n}_{pl}(t)\delta_{f}^{i}(\bar{T}_{jl}^{i}\tilde{n}_{p}^{l}(t)\tilde{x}_{p}^{j}(t)) \right. \\
&- \frac{1}{2} \left( \bar{T}_{jl}^{i} + \bar{T}_{jl}^{i} \right) \tilde{n}_{p}^{l}(t)\tilde{n}_{p}^{i}(t)\tilde{\psi}_{p(l}^{\beta} \bar{\psi}_{p\beta)}(t) \right\} \tilde{C}_{p}^{ri\gamma}(t) d\tilde{A}_{p}(t),
\end{align*} \]
where the symbol \( \mathcal{L} \) means the Lie derivative and \( \mathcal{L} \tilde{g}_{ij} \) is the Lie derivative of \( g_{ij} \) at \( \tilde{p} \).

Now putting
\[ \bar{S}_{p(r)}(t) = \tau \left\{ \tilde{n}_{pl}(t)\delta_{f}^{i}\mathcal{L} \tilde{g}_{ij} \tilde{n}_{p}^{j}(t) \tilde{x}_{p}^{l}(t) \right. \\
&- \frac{1}{2} \mathcal{L} \tilde{g}_{il} \tilde{n}_{p}^{l}(t)\tilde{n}_{p}^{j}(t) \tilde{\psi}_{p(l}^{\beta} \bar{\psi}_{p\beta)}(t) \right\} \tilde{C}_{p}^{ri\gamma}(t) d\tilde{A}_{p}(t), \]
we have
\[ \frac{(-1)^{r-1}}{(m-1)!} d\left( (\tilde{n}_{p}', \delta_{f}^{i} \tilde{n}_{p}, \cdots, \delta_{f}^{i} \tilde{n}_{p}, \tilde{d} x_{p}, \cdots, \tilde{d} x_{p}) \right) \\
(1.20)
= \frac{1}{m} \bar{H}_{p}^{r} \tilde{n}_{p} \delta_{f}^{i} \tau d\tilde{A}_{p}(t) \\
+ \frac{1}{\sqrt{\tilde{g}_{p}(t)}} \tilde{C}_{p}^{rs}(t) \tau_{r} \tau_{s} \left( \tilde{n}_{p} \delta_{f}^{i} \right)^{2} \sqrt{\tilde{g}_{p}(t)} d\tilde{A}_{p}(t) + \bar{S}_{p(r)}(t). \]
Since \( \tilde{n}_{pi}(t) \overline{\delta}_{\frac{1}{p}} \sqrt{\overline{g}^{*}(t)} \) at \( \bar{p} \) is independent with respect to \( t \), we can see the following

\[
\tilde{n}_{pi}(t) \overline{\delta}_{\frac{1}{p}} \sqrt{\overline{g}^{*}(t)} = \tilde{n}_{i} \overline{\delta}_{\frac{1}{i}} \overline{\sqrt{\overline{g}^{*}}},
\]

where \( \tilde{n}_{i} \) is the normal unit vector of \( \overline{W}^{m} \) at \( \overline{p} \) and \( \overline{g}^{*} \) is the determinant of the metric tensor \( \overline{g}_{ab} \) of \( \overline{W}^{m} \).

Integrating both members of (1.20) at \( \overline{p} \) over the interval \( 0 \leq t \leq 1 \), we obtain the following

\[
\frac{r(-1)^{r-1}}{(m-1)!} \int_{0}^{1} ((\tilde{n}_{p^{i}}, \delta_{1}, \delta_{n_{p}}, \cdots, \delta_{\tilde{n}_{p}}, d\tilde{x}_{p}, \cdots, d\tilde{x}_{p})) dt = m(\overline{H}_{r} - \overline{H}_{pr}(0)) \overline{n}_{i} \overline{\delta}_{\frac{1}{i}} \overline{\tau} d\overline{A}.
\]

(1.21)

where \( \overline{H}_{r} \) and \( \overline{H}_{pr}(0) \) are the \( r \)-th mean curvature of \( \overline{W}^{m} \) and \( \overline{W}^{m}_{p}(0) \) respectively, and \( d\overline{A} \) means the area element of \( \overline{W}^{m} \). Thus we can see that the left hand member of (1.21) is the exterior differential of the differential form attached to each point on the hypersurface \( W^{m} \). Let us denote a set of \( \overline{W}^{m}_{p}(t) \) for all \( p \in W^{m} \) by \( \overline{W}^{m}(t) \) and sets of the quantities of \( \overline{W}^{m}_{p}(t) \) by \( \overline{H}_{r}(t), \overline{C}_{(r)}^{\alpha \beta}(t), \overline{S}_{(r)}(t) \), etc.. Then integrating both members of (1.21) over \( W^{m} \) and applying Stokes' theorem, since \( W^{m} \) is closed, we have

\[
m \int_{\overline{W}^{m}} (H_{r} - \overline{H}_{r}(0)) \overline{n}_{i} \overline{\delta}_{\frac{1}{i}} \overline{\tau} d\overline{A} \]

(1.22)

\[
+ \int_{\overline{W}^{m}} \sqrt{\overline{g}^{*}} \int_{0}^{1} \overline{g}^{*}(t)^{-\frac{1}{2}} \overline{C}_{(r)}^{\alpha \beta}(t) dt \tau_{\alpha} \tau_{\beta} \overline{n}_{i} \overline{\delta}_{\frac{1}{i}} d\overline{A}
\]

\[
+ \int_{\overline{W}^{m}} \overline{S}_{(r)}(t) dt = 0.
\]

(1.22) is the integral form relating to the \( r \)-th mean curvature defined by two hypersurfaces \( W^{m} \) and \( \overline{W}^{m} \).

§ 2. A main theorem. We shall prove the following congruence theorem concerning the \( r \)-th mean curvature for closed hypersurfaces with the aid of the statements of the preceding section. We shall henceforth confine ourselves to two hypersurfaces \( W^{m} \) and \( \overline{W}^{m} \) which do not contain a piece of a hypersurface covered by the orbits of the transformations, which is expressed by \( f(x^{2}, \cdots, x^{m+1}) = 0 \).
Theorem 2.1. If the hypersurfaces $W^m$ and $\overline{W}^m$ in $S^{m+1}$ are closed orientable, and if there exists the relation

\[(2.1) \quad \overline{H}_r = \overline{H}_r(0)\]

at corresponding points along the orbits of the transformations, and if the following conditions are satisfied

\[(2.2) \quad \iint_{\mathbb{W}^m} \int_{0}^{1} \tilde{S}_{(r)} dt \geq 0\]

and that in case of $r \geq 2$, the second fundamental form of $\overline{W}_p^m(t)$ is positive definite for all $p \in W^m$ and all $t$ of the interval $0 \leq t \leq 1$, at the corresponding point $\overline{p}$, then $W^m$ and $\overline{W}^m$ are congruent mod $G$ to each other.

Proof. Using the condition that in case of $r \geq 2$, the second fundamental form of $\overline{W}_p^m(t)$ is positive definite at $\overline{p}$, for all $r$, we have that the quantity

$$\sqrt{\overline{g}^*} \int_{0}^{1} \tilde{g}^* (t) \overline{C}_{(r)}^{*l} dt \tau_{\alpha} \tau_{\beta} (\overline{n}_i \delta_{1}^{i})^{2} d\overline{A} = 0$$

From the hypersurfaces $W^m$ and $\overline{W}^m$ do not contain a piece of a hypersurface covered by the orbits of transformations, that is, a point on $\overline{W}^m$ such that $\overline{n}_i \delta_{1}^{i} = 0$ must be an isolate point. Moreover since $\tau$ is a continuous function of $\overline{W}^m$, we conclude

$$\tau = \text{constant}$$

at every point of $\overline{W}^m$. This fact shows that $W^m$ and $\overline{W}^m$ are congruent mod $G$.

Remark 1. If $G$ is a group of isometric transformations, then we have that $\overline{S}_{(r)}(t) = 0$ and $\overline{H}_r(0) = H_r$, etc., and Theorem 2.1 coincide with the theorem given in the introduction.

Remark 2. In case of $r = 1$, that is, the first mean curvature, we can cancel the assumption that our space $S^{m+1}$ is of constant curvature.
References


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