A semi-group formula for the Riesz potentials

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Abstract. The purpose of this article is to establish a semi-group formula for the Riesz potentials of $L^p$-functions. As preparations, we study the Lizorkin space $\Phi(\mathbb{R}^n)$ and investigate integral estimates of the Riesz potentials of functions in the spaces $L^{p,r,s}(\mathbb{R}^n)$.

Key words: Riesz potentials, Lizorkin space, semi-group formula.

1. Introduction

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. Throughout this paper let $0 < \alpha < \infty$ and $1 < p < \infty$. For real numbers $r$ and $s$ we define the spaces $L^{p;r,s}(\mathbb{R}^n)$ as follows:

$$
L^{p;r,s}(\mathbb{R}^n) = \left\{ f : \| f \|_{p;r,s} = \left( \int_{\mathbb{R}^n} |f(x)|^p |x|^r (1 + |\log |x||)^s p \, dx \right)^{1/p} < \infty \right\}.
$$

We simply write $L^{p;0,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $\|f\|_{p;0,0} = \|f\|_p$. Let $G_\alpha(x)$ be the Bessel kernel of order $\alpha$ defined by

$$
G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2} \Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta} e^{-\delta/(4\pi)} \delta^{(\alpha-n)/2} \frac{d\delta}{\delta}.
$$

Since the Bessel kernel $G_\alpha(x)$ is integrable ([St, Proposition 2 in Chap. V]), for $f \in L^p(\mathbb{R}^n)$ the Bessel potential of order $\alpha$ of $f$

$$
G_\alpha f(x) = \int G_\alpha(x - y) f(y) \, dy
$$

belongs to $L^p(\mathbb{R}^n)$. For the Bessel potentials, it is known that the following semi-group formula holds ([St, 3.3 in Chap. V]):

$$
G_{\alpha + \beta} f = G_\alpha (G_\beta f), \quad f \in L^p(\mathbb{R}^n).
$$

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The purpose of this article is to establish a semi-group formula for the Riesz potentials of $L^p$-functions. Let $N$ be the set of nonnegative integers and $2N$ stands for the set of nonnegative even numbers. The Riesz kernel $\kappa_\alpha(x)$ of order $\alpha$ is given by

$$
\kappa_\alpha(x) = \frac{1}{\gamma_{\alpha,n}} \begin{cases} 
|x|^{\alpha-n}, & \alpha - n \notin 2N \\
(\delta_{\alpha,n} - \log |x|)|x|^{\alpha-n}, & \alpha - n \in 2N 
\end{cases}
$$

with

$$
\gamma_{\alpha,n} = \begin{cases} 
\pi^{n/2}2^{\alpha}\Gamma(\alpha/2)/\Gamma((n-\alpha)/2), & \alpha - n \notin 2N \\
(-1)^{(\alpha-n)/2}2^{\alpha-1}\pi^{n/2}\Gamma(\alpha/2)((\alpha-n)/2)!, & \alpha - n \in 2N 
\end{cases}
$$

and

$$
\delta_{\alpha,n} = \frac{\Gamma'(\alpha/2)}{2\Gamma(\alpha/2)} + \frac{1}{2} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{(\alpha-n)/2} - C \right) - \log \pi
$$

where $C$ is Euler’s constant. For a function $f$ we define the Riesz potential $U_\alpha f$ of order $\alpha$ of $f$ as follows:

$$
U_\alpha f(x) = \int \kappa_\alpha(x-y)f(y)dy
$$

if it exists. If $\alpha - (n/p) < 0$, then for $f \in L^p(\mathbb{R}^n)$, $U_\alpha f$ exists and satisfies the following inequality ([SW, Theorem B*]):

$$
\|U_\alpha f\|_{p,-\alpha,0} \leq C\|f\|_p.
$$

However, if $\alpha - (n/p) \geq 0$, then for an $L^p$-function $f$, $U_\alpha f$ does not necessarily exist. To consider the Riesz potentials of $L^p$-functions we introduce the Riesz kernels of type $(\alpha, k)$. For an integer $k$ we set

$$
\kappa_{\alpha,k}(x,y) = \kappa_\alpha(x-y) - \sum_{|\gamma| \leq k} \frac{x^\gamma}{\gamma!} D^\gamma \kappa_\alpha(-y)
$$

where we regard the second term of the right-hand side as zero if $k \leq -1$, and $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index, $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ ($x = (x_1, \ldots, x_n)$),
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\[ D^\gamma = D_1^{\gamma_1} \ldots D_n^{\gamma_n} \quad (D_j = \partial/\partial x_j), \quad \gamma! = \gamma_1! \ldots \gamma_n! \quad \text{and} \quad |\gamma| = \gamma_1 + \cdots + \gamma_n. \]

We also denote

\[ p_{\alpha,k}(x, y) = -\sum_{|\gamma| \leq k} x^\gamma D^\gamma \kappa_\alpha(-y). \]

For a function \( f \) we define the Riesz potential \( U_{\alpha,k}f \) and the Riesz polynomial \( P_{\alpha,k}f \) of type \((\alpha, k)\) of \( f \) as follows:

\[
U_{\alpha,k}f(x) = \int \kappa_{\alpha,k}(x, y)f(y)dy, \quad P_{\alpha,k}f(x) = \int p_{\alpha,k}(x, y)f(y)dy
\]

if they exist. The Riesz polynomial \( P_{\alpha,k}f \) is a polynomial of degree \( k \) if it exists.

Our plan is as follows. In Section 2 we introduce and study the Lizorkin space \( \Phi(\mathbb{R}^n) \). The Lizorkin space \( \Phi(\mathbb{R}^n) \) has been studied by several authors (cf. [Sa], [SKM]). It is known that \( \Phi(\mathbb{R}^n) \) is invariant with respect to the Riesz potential operator and a semi-group formula for the Riesz potentials of functions in \( \Phi(\mathbb{R}^n) \) holds. We establish the fact that certain subspaces of \( \Phi(\mathbb{R}^n) \) are dense in \( L^p(\mathbb{R}^n) \) (Proposition 2.9). In Section 3 we give integral estimates for the Riesz potentials of type \((\alpha, k)\) of functions in the spaces \( L^{p,r,s}(\mathbb{R}^n) \) (Theorem 3.2 and Corollary 3.8). In particular, it turns out that for a function \( f \in L^{p,r,s}(\mathbb{R}^n) \), \( U_{\alpha,k}f \) exists if \( r > -n/p' \) and \( \alpha + r - (n/p) \notin \mathbb{N} \) where \( k \) is the integral part of \( \alpha + r - (n/p) \). In Section 4 we prove a semi-group formula for the Riesz potentials of \( L^p \)-functions (Theorem 4.4). Throughout this paper we use the symbol \( C \) for a generic positive constant whose value may be different at each occurrence.

2. The Lizorkin space \( \Phi(\mathbb{R}^n) \)

We denote the Schwartz space on \( \mathbb{R}^n \) by \( \mathcal{S}(\mathbb{R}^n) \). That is, \( \mathcal{S}(\mathbb{R}^n) \) is the space of all \( C^\infty \)-functions \( \varphi \) in \( \mathbb{R}^n \) such that

\[
q_{\gamma,\delta}(\varphi) = \sup_{x \in \mathbb{R}^n} |x^\gamma D^\delta \varphi(x)| < \infty
\]

for all multi-indices \( \gamma \) and \( \delta \). The space \( \mathcal{S}(\mathbb{R}^n) \) is a Fréchet space with a countable family of semi-norms \( \{q_{\gamma,\delta}\} \). For a function \( f \in \mathcal{S}(\mathbb{R}^n) \) the Riesz potential \( U_{\alpha}f(x) \) exists for any \( x \in \mathbb{R}^n \). Moreover in case \( k < \alpha \), \( U_{\alpha,k}f(x) \)
and \( P_{\alpha,k}f(x) \) exist for any \( x \in \mathbb{R}^n \) and
\[
U_{\alpha,k}f(x) = U_\alpha f(x) + P_{\alpha,k}f(x).
\]

The Lizorkin space \( \Phi(\mathbb{R}^n) \) is defined by
\[
\Phi(\mathbb{R}^n) = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \int \varphi(x)x^\gamma dx = 0 \quad \text{for all } \gamma \right\}
\]
([SKM, Section 25 in Chap. 5]). Further, we introduce the space \( \Psi(\mathbb{R}^n) \) as follows:
\[
\Psi(\mathbb{R}^n) = \left\{ \psi \in \mathcal{S}(\mathbb{R}^n) : D^\gamma \psi(0) = 0 \quad \text{for all } \gamma \right\}.
\]

The Fourier transform \( \mathcal{F}f \) and the inverse Fourier transforms \( \mathcal{F}^{-1}f \) of an integrable function \( f \) are defined by
\[
\mathcal{F}f(x) = \int e^{-ix \cdot y} f(y)dy, \quad \mathcal{F}^{-1}f(x) = \int e^{ix \cdot y} f(y)dy = \mathcal{F}f(-x)
\]
where \( x \cdot y = x_1y_1 + \cdots + x_ny_n \). By the Fourier inversion formula, for \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) we have the equality
\[
\mathcal{F}\mathcal{F}\varphi = \mathcal{F}\mathcal{F}\varphi = (2\pi)^n \varphi.
\]

Noting that
\[
D^\gamma(\mathcal{F}\varphi)(0) = \int \varphi(y)(-iy)^\gamma dy
\]
and
\[
\int \mathcal{F}\psi(y)(iy)^\gamma dy = (2\pi)^n D^\gamma \psi(0)
\]
for \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \), we see that
\[
\Phi(\mathbb{R}^n) = \mathcal{F}(\Psi(\mathbb{R}^n)), \quad \Psi(\mathbb{R}^n) = \mathcal{F}(\Phi(\mathbb{R}^n)).
\]
topological dual space of $S(\mathbb{R}^n)$. We use the notation $\langle u, \varphi \rangle$ for the canonical bilinear form on $S'(\mathbb{R}^n) \times S(\mathbb{R}^n)$. For $u \in S'(\mathbb{R}^n)$ we define the Fourier transform $\mathcal{F}u$ (resp. the inverse Fourier transform $\mathcal{F}^{-1}u$) to be the element of $S'(\mathbb{R}^n)$ whose value at $\varphi \in S(\mathbb{R}^n)$ is $\langle \mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}^{-1}\varphi \rangle$ (resp. $\langle \mathcal{F}^{-1}u, \varphi \rangle = \langle u, \mathcal{F}\varphi \rangle$). The Fourier transform of the Riesz kernel $\kappa_\alpha \in S'(\mathbb{R}^n)$ is given by

$$F\kappa_\alpha(x) = (2\pi)^\alpha \text{Pf.}|x|^{-\alpha},$$

where Pf. stands for the pseudo function ([Sc, Section 4 in Chap VII]). We note that for $\psi \in \Psi(\mathbb{R}^n)$

$$\langle \text{Pf.}|x|^{-\alpha}, \psi \rangle = \int |x|^{-\alpha}\psi(x)dx.$$  

(2.7)

The Lizorkin space $\Phi(\mathbb{R}^n)$ has the following properties.

**Proposition 2.1** ([SKM, Theorem 25.1], [Sa, Theorem 2.16]) For $\varphi \in \Phi(\mathbb{R}^n)$, $U_\alpha \varphi$ belongs to $\Phi(\mathbb{R}^n)$ and

$$U_{\alpha+\beta}\varphi = U_\alpha(U_\beta\varphi).$$

**Proposition 2.2** ([Sa, Theorem 2.7]) The Lizorkin space $\Phi(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

We establish that not only the space $\Phi(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, but also certain subspaces of $\Phi(\mathbb{R}^n)$ are dense in $L^p(\mathbb{R}^n)$. For $\alpha > 0$ and a nonnegative integer $k$ with $k < \alpha$, the space $\Phi_{\alpha,k}(\mathbb{R}^n)$ is defined by

$$\Phi_{\alpha,k}(\mathbb{R}^n) = \left\{ \varphi \in \Phi(\mathbb{R}^n) : \int \varphi(x)D^\gamma \kappa_\alpha(x)dx = 0 \text{ for } |\gamma| \leq k \right\}.$$  

In the remainder of this section we prove that if $\alpha - (n/p) \notin \mathbb{N}$, then the space $\Phi_{\alpha,\lfloor\alpha-(n/p)\rfloor}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ where $\lfloor\alpha-(n/p)\rfloor$ is the integral part of $\alpha - (n/p)$.

To prove the above fact we prepare five lemmas and one remark.

The first lemma is proved by the similar way to [Ku, Lemma 2.2].

**Lemma 2.3** For a nonnegative integer $k$ there exists a function $\theta(t) \in \Phi(R^1)$ such that
\[ D^i \theta(0) = \begin{cases} 1, & i = 0 \\ 0, & i = 1, \ldots, k \end{cases} \] (2.8)

where \( D^i \theta \) is the derivative of order \( i \) of \( \theta \).

**Lemma 2.4** For a nonnegative integer \( k \) there exists a function \( \zeta(x) \in \Phi(\mathbb{R}^n) \) such that

\[ D^\delta \zeta(0) = \begin{cases} 1, & \delta = 0 \\ 0, & 0 < |\delta| \leq k. \end{cases} \] (2.9)

**Proof.** By Lemma 2.3 there exists \( \theta(t) \in \Phi(\mathbb{R}^1) \) which satisfies (2.8). We put \( \zeta(x) = \theta(x_1) \ldots \theta(x_n) \). It is clear that \( \zeta \in \Phi(\mathbb{R}^n) \). Moreover we have

\[ \zeta(0) = \theta(0) \ldots \theta(0) = 1 \]

and for \( 0 < |\delta| \leq k \)

\[ D^\delta \zeta(0) = D^{\delta_1} \theta(0) \ldots D^{\delta_n} \theta(0) = 0 \]

because there exists \( i \) such that \( \delta_i \neq 0 \). Thus we obtain the lemma. \( \square \)

**Lemma 2.5** For a nonnegative integer \( k \) there exist functions \( \{ \zeta_\gamma \}_{|\gamma| \leq k} \subset \Phi(\mathbb{R}^n) \) such that

\[ D^\delta \zeta_\gamma(0) = \begin{cases} 1, & \delta = \gamma \\ 0, & \delta \neq \gamma \end{cases} \] (2.10)

for \( |\delta|, |\gamma| \leq k \).

**Proof.** By Lemma 2.4 there exists a function \( \zeta \in \Phi(\mathbb{R}^n) \) which satisfies (2.9). For \( |\gamma| \leq k \) we put

\[ \zeta_\gamma(x) = \omega_\gamma(x) \zeta(x) \]

where \( \omega_\gamma(x) = x^\gamma / \gamma! \). It is clear that \( \zeta_\gamma \in \Phi(\mathbb{R}^n) \) for \( |\gamma| \leq k \). We prove (2.10). By Leipniz’s formula we have

\[ D^\delta \zeta_\gamma(x) = D^\delta (\omega_\gamma(x) \zeta(x)) = \sum_{\eta \leq \delta} \binom{\delta}{\eta} D^n \omega_\gamma(x) D^{\delta-\eta} \zeta(x) \]
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where

$$\left( \begin{array}{c} \delta \\ \eta \end{array} \right) = \left( \begin{array}{c} \delta_1 \\ \eta_1 \end{array} \right) \ldots \left( \begin{array}{c} \delta_n \\ \eta_n \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} \delta_i \\ \eta_i \end{array} \right) = \frac{\delta_i!}{\eta_i!(\delta_i - \eta_i)!}.$$ 

Since

$$D^\eta \omega_\gamma(x) = \begin{cases} \omega_{\gamma - \eta}(x), & \eta \leq \gamma, \\ 0, & \text{otherwise}, \end{cases}$$

we see that

$$D^\delta \zeta_\gamma(0) = \sum_{\eta \leq \min(\delta, \gamma)} \left( \begin{array}{c} \delta \\ \eta \end{array} \right) \omega_{\gamma - \eta}(0) D^{\delta - \eta} \zeta(0)$$

where \(\min(\delta, \gamma) = (\min(\delta_1, \gamma_1), \ldots, \min(\delta_n, \gamma_n)).\) In case of \(\delta = \gamma,\) by (2.9) and the fact that

$$\omega_\gamma(0) = \begin{cases} 1, & \gamma = 0 \\ 0, & \gamma \neq 0, \end{cases} \quad (2.11)$$

we have

$$D^\delta \zeta_\gamma(0) = D^\gamma \zeta_\gamma(0) = \sum_{\eta \leq \gamma} \left( \begin{array}{c} \gamma \\ \eta \end{array} \right) \omega_{\gamma - \eta}(0) D^{\gamma - \eta} \zeta(0) = \left( \begin{array}{c} \gamma \\ \gamma \end{array} \right) \omega_0(0) \zeta(0) = 1.$$ 

Next, let \(\delta \neq \gamma.\) There are two cases. Firstly we consider the case that there exists \(i\) such that \(\delta_i < \gamma_i.\) For \(\eta \leq \min(\delta, \gamma)\) we obtain that \(\eta_i \leq \delta_i < \gamma_i,\) and hence \(\gamma - \eta > 0.\) Therefore by (2.11) \(\omega_{\gamma - \eta}(0) = 0\) for \(\eta \leq \min(\delta, \gamma),\) and hence \(D^\delta \zeta_\gamma(0) = 0.\) Secondly we consider the case that there exists \(i\) such that \(\delta_i > \gamma_i.\) For \(\eta \leq \min(\delta, \gamma)\) we obtain that \(\eta_i \leq \gamma_i < \delta_i,\) and hence \(\delta - \eta > 0.\) Therefore by (2.9) \(D^{\delta - \eta} \zeta(0) = 0\) for \(\eta \leq \min(\delta, \gamma),\) and hence \(D^\delta \zeta_\gamma(0) = 0.\) Consequently, we see that \(D^\delta \zeta_\gamma(0) = 0\) for \(\delta \neq \gamma.\) Thus we obtain (2.10) and complete the proof of the lemma.

Lemma 2.6 For \(\alpha > 0\) and a nonnegative integer \(k\) with \(k < \alpha,\) there exist functions \(\{\mu_\gamma\}_{|\gamma| \leq k} \subset \Phi(\mathbb{R}^n)\) such that

$$\int \mu_\gamma(x) D^\delta \kappa_\alpha(x) dx = \begin{cases} 1, & \gamma = \delta \\ 0, & \gamma \neq \delta \end{cases}$$
for $|\gamma|, |\delta| \leq k$.

**Proof.** By the previous lemma there exist functions $\{\zeta_\gamma\}_{|\gamma| \leq k} \subset \Phi(\mathbb{R}^n)$ which satisfy (2.10). We put

$$\mu_\gamma(x) = \frac{(-1)^{|\gamma|}}{(2\pi)^{\alpha+n}} \mathcal{F}(|\xi|^\alpha \mathcal{F}\zeta_\gamma(\xi))(x)$$

for $|\gamma| \leq k$. Since $\zeta_\gamma \in \Phi(\mathbb{R}^n)$, by (2.5) we see that $\mathcal{F}\zeta_\gamma(\xi) \in \Psi(\mathbb{R}^n)$, $|\xi|^\alpha \mathcal{F}\zeta_\gamma(\xi) \in \Psi(\mathbb{R}^n)$ and $\mu_\gamma(x) \in \Phi(\mathbb{R}^n)$. Since $\kappa_\alpha \in \mathcal{S}'(\mathbb{R}^n)$ and $\mu_\gamma \in \Phi(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, we can consider

$$I = \frac{1}{(2\pi)^n} \langle \mathcal{F}(D^\delta \kappa_\alpha), \mathcal{F}\mu_\gamma \rangle.$$  

By (2.2) we have

$$I = \frac{1}{(2\pi)^n} \langle D^\delta \kappa_\alpha, \mathcal{F}\mathcal{F}\mu_\gamma \rangle = \langle D^\delta \kappa_\alpha, \mu_\gamma \rangle.$$

Moreover, since $D^\delta \kappa_\alpha(x)$ is locally integrable and $D^\delta \kappa_\alpha(x)\mu_\gamma(x)$ is integrable for $|\delta| < \alpha$, we have

$$I = \int D^\delta \kappa_\alpha(x)\mu_\gamma(x)dx \quad (2.12)$$

for $|\delta|, |\gamma| \leq k(< \alpha)$. On the other hand, since $\mathcal{F}\kappa_\alpha(\xi) = (2\pi)^{\alpha} \text{Pf.}|\xi|^{-\alpha}$ in $\mathcal{S}'(\mathbb{R}^n)$ by (2.6), we have

$$I = \frac{(2\pi)^{\alpha}(-i)^{|\delta|}}{(2\pi)^n} \langle \xi^\delta \text{Pf.}|\xi|^{-\alpha}, \mathcal{F}\mu_\gamma \rangle$$

$$= \frac{(2\pi)^{\alpha}(-i)^{|\delta|}}{(2\pi)^n} \left\langle \xi^\delta \text{Pf.}|\xi|^{-\alpha}, \frac{(-1)^{|\gamma|}}{(2\pi)^{\alpha+n}} \mathcal{F}(|\xi|^\alpha \mathcal{F}\zeta_\gamma(\xi)) \right\rangle$$

$$= \frac{(-i)^{|\delta|}}{(2\pi)^n} \langle \xi^\delta \text{Pf.}|\xi|^{-\alpha}, (\xi^\delta \mathcal{F}\zeta_\gamma(\xi)) \rangle$$

$$= \frac{(-i)^{|\delta|}}{(2\pi)^n} \langle \text{Pf.}|\xi|^{-\alpha}, (\xi^\delta \mathcal{F}\zeta_\gamma(\xi)) \rangle.$$
Since \((-1)^{|\gamma|} \xi^\delta |\xi|^\alpha \mathcal{F} \zeta_\gamma (\xi) \in \Psi (\mathbb{R}^n)\), by (2.7) and (2.10) we obtain that

\[
I = \frac{(-i)^{|\delta|}}{(2\pi)^n} \int |\xi|^{-\alpha} (-1)^{|\gamma|} \xi^\delta |\xi|^\alpha \mathcal{F} \zeta_\gamma (\xi) d\xi
\]

\[
= \frac{(-i)^{|\delta|}(-1)^{|\gamma|}}{(2\pi)^n i^{|\delta|}} \int (i\xi)^{\delta} \mathcal{F} \zeta_\gamma (\xi) d\xi
\]

\[
= \frac{(-1)^{|\delta|+|\gamma|}}{(2\pi)^n} D^\delta (\mathcal{F} \mathcal{F} \zeta_\gamma) (0) = (-1)^{|\delta|+|\gamma|} D^\delta \zeta_\gamma (0)
\]

\[
= \begin{cases} 
1, & \delta = \gamma \\
0, & \delta \neq \gamma
\end{cases}
\] (2.13)

for \(|\delta|, |\gamma| \leq k\). By (2.12) and (2.13) we get

\[
\int \mu_{\gamma}(x) D^\delta \kappa_\alpha (x) dx = \begin{cases} 
1, & \delta = \gamma \\
0, & \delta \neq \gamma
\end{cases}
\]

for \(|\delta|, |\gamma| \leq k\). Thus we obtain the lemma. \(\Box\)

Here, we remark the following fact.

**Remark 2.7** Let \(H(x) = |x|^{2\ell} \log |x|\) where \(\ell\) is a nonnegative integer. Then

\[
D^\delta H(x) = \begin{cases} 
P(x) \log |x| + Q(x), & |\delta| \leq 2\ell \\
Q(x), & |\delta| \geq 2\ell + 1
\end{cases}
\]

where \(P(x)\) is a homogeneous polynomial of degree \(2\ell - |\delta|\) and \(Q(x)\) is a homogeneous function of degree \(2\ell - |\delta|\).

**Lemma 2.8** Let \(\alpha - (n/p) > 0\) and \(\alpha - (n/p) \notin \mathbb{N}\). Then there exist functions \(\{\mu_{\gamma,m}\}_{|\gamma| \leq [\alpha - (n/p)], m=1,2,...} \subset \Phi (\mathbb{R}^n)\) such that

(i) for \(|\gamma|, |\delta| \leq [\alpha - (n/p)]\)

\[
\int \mu_{\gamma,m} (x) D^\delta \kappa_\alpha (x) dx = \begin{cases} 
1, & \gamma = \delta \\
0, & \gamma \neq \delta
\end{cases}
\]

and
(ii) for $|\gamma| \leq [\alpha - (n/p)]$

$$\|\mu_{\gamma,m}\|_p \to 0 \quad (m \to \infty).$$

Proof. Since $[\alpha - (n/p)] < \alpha$, by Lemma 2.6 there exist functions 
\(
\{\mu_{\gamma}\}_{|\gamma| \leq [\alpha - (n/p)]} \subset \Phi(\mathbb{R}^n)
\)
such that

$$\int \mu_{\gamma}(x)D^\delta \kappa_\alpha(x)dx = \begin{cases} 
1, & \gamma = \delta \\
0, & \gamma \neq \delta 
\end{cases} \quad (2.14)$$

for $|\gamma|, |\delta| \leq [\alpha - (n/p)]$. We put

$$\mu_{\gamma,m}(x) = \frac{1}{m^{\alpha - |\gamma|}} \mu_{\gamma}\left(\frac{x}{m}\right)$$

for $|\gamma| \leq [\alpha - (n/p)]$ and $m = 1, 2, \ldots$. It is clear that $\mu_{\gamma,m} \in \Phi(\mathbb{R}^n)$. First we consider the case that $\alpha - n$ is a nonnegative even number. In this case $D^\delta \kappa_\alpha(x)$ is a homogeneous function of degree $\alpha - n - |\delta|$. Hence for $|\gamma|, |\delta| \leq [\alpha - (n/p)]$, by the change of variables we have

$$\int \mu_{\gamma,m}(x)D^\delta \kappa_\alpha(x)dx$$

$$= \frac{1}{m^{\alpha - |\gamma|}} \int \mu_{\gamma}\left(\frac{x}{m}\right)D^\delta \kappa_\alpha(x)dx = m^{\gamma - \alpha + n} \int \mu_{\gamma}(y)D^\delta \kappa_\alpha(my)dy$$

$$= m^{\gamma - |\delta|} \int \mu_{\gamma}(y)D^\delta \kappa_\alpha(y)dy = \begin{cases} 
1, & \gamma = \delta \\
0, & \gamma \neq \delta 
\end{cases}$$

on account of (2.14). Next we consider the case that $\alpha - n$ is a nonnegative even number. In this case, since $[\alpha - (n/p)] \leq \alpha - n$, by Remark 2.7 for $|\delta| \leq [\alpha - (n/p)]$

$$D^\delta \kappa_\alpha(x) = P(x) \log|x| + Q(x)$$

where $P(x)$ is a homogeneous polynomial of degree $\alpha - n - |\delta|$ and $Q(x)$ is a homogeneous function of degree $\alpha - n - |\delta|$. Hence for $|\gamma|, |\delta| \leq [\alpha - (n/p)]$, we have
\[
\int \mu_{\gamma,m}(x)D^\delta \kappa_{\alpha}(x)dx \\
= \frac{1}{m^{\alpha-|\gamma|}} \int \mu_{\gamma}\left(\frac{x}{m}\right)D^\delta \kappa_{\alpha}(x)dx \\
= m^{\gamma|-\alpha+n} \int \mu_{\gamma}(y)D^\delta \kappa_{\alpha}(my)dy \\
= m^{\gamma|-\alpha+n} \int \mu_{\gamma}(y)(P(my)\log(m|y|) + Q(my))dy \\
= m^{|\gamma|-\alpha+n} \int (\mu_{\gamma}(y)(m^{\alpha-n-|\delta|} P(y)(\log m + \log |y|) + m^{\alpha-n-|\delta|} Q(y))dy \\
= m^{|\gamma|-|\delta|} \left( \int \mu_{\gamma}(y)(P(y)\log |y| + Q(y))dy + \log m \int \mu_{\gamma}(y)P(y)dy \right) \\
= m^{|\gamma|-|\delta|} \int \mu_{\gamma}(y)D^\delta \kappa_{\alpha}(y)dy
\]

because \( P(y) \) is a polynomial and \( \mu_{\gamma} \in \Phi(\mathbb{R}^n) \). Therefore, by (2.14)

\[
\int \mu_{\gamma,m}(x)D^\delta \kappa_{\alpha}(x)dx = \begin{cases} 1, & \gamma = \delta \\ 0, & \gamma \neq \delta \end{cases}
\]

for \(|\gamma|,|\delta| \leq [\alpha - (n/p)]\). Thus we obtain (i). Further, by the change of variables we have

\[
\|\mu_{\gamma,m}\|_p = \left( \int |\mu_{\gamma,m}(x)|^p dx \right)^{1/p} = \left( \int \frac{1}{m^{(\alpha-|\gamma|)p}} \left| \mu_{\gamma}\left(\frac{x}{m}\right) \right|^p dx \right)^{1/p} \\
= m^{(n/p)-\alpha+|\gamma|} \left( \int |\mu_{\gamma}(y)|^p dy \right)^{1/p}
\]

Since \( \alpha - (n/p) \notin \mathbb{N} \), the condition \(|\gamma| \leq [\alpha - (n/p)]\) implies \((n/p) - \alpha + |\gamma| < 0\), and hence \(\|\mu_{\gamma,m}\|_p \to 0 \ (m \to \infty)\) for \(|\gamma| \leq [\alpha - (n/p)]\). This shows (ii).

Thus we complete the proof of the lemma.

Now we prove the denseness of \( \Phi_{\alpha,[\alpha-(n/p)]}(\mathbb{R}^n) \) in \( L^p(\mathbb{R}^n) \).

**Proposition 2.9** Let \( \alpha - (n/p) \notin \mathbb{N} \). Then the space \( \Phi_{\alpha,[\alpha-(n/p)]}(\mathbb{R}^n) \) is dense in \( L^p(\mathbb{R}^n) \).
Proof. Since \( \Phi(\mathbb{R}^n) \) is dense in \( L^p(\mathbb{R}^n) \) by Proposition 2.2, it is sufficient to show that \( \Phi_{\alpha,[\alpha-(n/p)]}(\mathbb{R}^n) \) is dense in \( \Phi(\mathbb{R}^n) \) with respect to \( L^p(\mathbb{R}^n) \)-norm. In case \( \alpha - (n/p) < 0 \), \( \Phi_{\alpha,[\alpha-(n/p)]}(\mathbb{R}^n) = \Phi(\mathbb{R}^n) \), and hence the assertion is obvious. Let \( \alpha - (n/p) > 0 \). Then there exist functions \( \{\mu_{\gamma,m}\}_{|\gamma| \leq \alpha-(n/p)}, m=1,2,... \subset \Phi(\mathbb{R}^n) \) which satisfy (i) and (ii) in Lemma 2.8. For \( \varphi \in \Phi(\mathbb{R}^n) \) we put

\[
0\varphi_m(x) = \varphi(x) - \sum_{|\delta| \leq \alpha-(n/p)} \left( \int \varphi(y) D^\delta \kappa_\alpha(y) dy \right) \mu_{\delta,m}(x).
\]

It is clear that \( \varphi_m \in \Phi(\mathbb{R}^n) \). Moreover for \( |\gamma| \leq \alpha-(n/p) \), by (i) in Lemma 2.8 we have

\[
\int \varphi_m(x) D^\gamma \kappa_\alpha(x) dx = \int \varphi(x) D^\gamma \kappa_\alpha(x) dx - \sum_{|\delta| \leq \alpha-(n/p)} \left( \int \varphi(y) D^\delta \kappa_\alpha(y) dy \right) \left( \int \mu_{\delta,m}(x) D^\gamma \kappa_\alpha(x) dx \right) = 0.
\]

Hence \( \varphi_m \in \Phi_{\alpha,[\alpha-(n/p)]}(\mathbb{R}^n) \). Further, by (ii) in Lemma 2.8 we obtain

\[
\|\varphi_m - \varphi\|_p \leq \sum_{|\delta| \leq \alpha-(n/p)} \left\| \int \varphi(y) D^\delta \kappa_\alpha(y) dy \right\|_{\mu_{\delta,m}} \rightarrow 0 \quad (m \rightarrow \infty).
\]

Namely, \( \varphi_m \) converges to \( \varphi \) with respect to \( L^p(\mathbb{R}^n) \)-norm as \( m \rightarrow \infty \). Thus \( \Phi_{\alpha,[\alpha-(n/p)]}(\mathbb{R}^n) \) is dense in \( \Phi(\mathbb{R}^n) \) with respect to \( L^p(\mathbb{R}^n) \)-norm. This completes the proof of Proposition 2.9. \( \square \)

3. Riesz potentials on the spaces \( L^{p,r,s}(\mathbb{R}^n) \)

As defined in section 1, for \( p > 1 \) and \( r, s \in \mathbb{R} \) the spaces \( L^{p,r,s}(\mathbb{R}^n) \) are given by

\[
L^{p,r,s}(\mathbb{R}^n) = \left\{ \varphi \in L^p(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\varphi(x)|^r dx < \infty \right\}.
\]

For \( \alpha > 0 \) and \( \gamma \in \mathbb{R} \), the Riesz potential of order \( \alpha \) with respect to \( \gamma \) is defined by

\[
\mathcal{R}_\gamma \Phi_{\alpha}(\varphi)(x) = \frac{1}{\Gamma(\frac{\gamma}{\alpha})} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n/\alpha - \gamma/\alpha}} dy.
\]
\[ L^{p;r,s}(\mathbb{R}^n) = \left\{ f : \|f\|_{p;r,s} = \left( \int_{\mathbb{R}^n} |f(x)||x|^{r p} (1 + |\log |x||)^{s p} dx \right)^{1/p} < \infty \right\}. \]

In this section we investigate integral estimates for Riesz potentials of functions in \( L^{p;r,s}(\mathbb{R}^n) \). To do so we introduce a kernel \( K_{\alpha,\ell}(x) = K_\alpha(x)(1 + \log |x|)^{\ell} \) (\( \alpha > 0, \ell \in \mathbb{N} \)) where \( K_\alpha(x) \) is a homogeneous function of degree \( \alpha - n \) which is infinitely differentiable in \( \mathbb{R}^n \setminus \{0\} \). For multi-index \( \gamma \) we see that
\[ D^\gamma K_{\alpha,\ell}(x) = \sum_{j=0}^{\min(|\gamma|,\ell)} H_{\gamma,j}(x)(1 + \log |x|)^{\ell - j} \]
where \( H_{\gamma,j}(x) \) is a homogeneous function of degree \( \alpha - n - |\gamma| \). Hence
\[ |D^\gamma K_{\alpha,\ell}(x)| \leq C|x|^{\alpha - n - |\gamma|}(1 + |\log |x||)^{\ell}. \quad (3.1) \]

Further, for an integer \( k \) we set
\[ K_{\alpha,\ell,k}(x,y) = K_{\alpha,\ell}(x-y) - \sum_{|\gamma| \leq k} \frac{x^\gamma}{\gamma!} D^\gamma K_{\alpha,\ell}(-y) \]
where we regard the second term of the right-hand side as zero if \( k \leq -1 \). For \( x \in \mathbb{R}^n \) we put \( \ell_x = \{tx : 0 \leq t \leq 1\} \) and denote by \( d(y,\ell_x) \) the distance between \( y \) and \( \ell_x \).

**Lemma 3.1** Let \( k \) be a nonnegative integer. Then for \( d(y,\ell_x) > |x|/2 \)
\[ |K_{\alpha,\ell,k}(x,y)| \leq C|x|^{k+1}|y|^{|\alpha - n - k - 1|}(1 + |\log |y||)^{\ell}. \]

**Proof.** Let \( x = 0 \). Then \( d(y,\ell_x) > |x|/2 \) means \( y \neq 0 \). For \( y \neq 0 \) we see that
\[ K_{\alpha,\ell,k}(0,y) = K_{\alpha,\ell}(-y) - K_{\alpha,\ell}(-y) = 0, \]
and the right-hand side of the required inequality is zero. Hence the lemma holds. Let \( x \neq 0 \). We note that \( K_{\alpha,\ell}(z-y) \) is a \( C^\infty \)-function as a function of \( z \) in \( \mathbb{R}^n \setminus \{y\} \). Therefore, for \( d(y,\ell_x) > |x|/2, K_{\alpha,\ell}(z-y) \) is a \( C^\infty \)-function as a function of \( z \) in the open set \( U_x = \{z : d(z,\ell_x) < |x|/2\} \). Noting that
$\ell_x \subset U_x$ for $z \in U_x$ and $U_x$ contains 0, we apply the integral remainder formula for Taylor’s theorem to $K_{\alpha,\ell}(z - y)$ in $U_x$. Then we get

$$K_{\alpha,\ell}(z - y) = \sum_{|\gamma| \leq k} \frac{z^\gamma}{\gamma!} D^\gamma K_{\alpha,\ell}(-y) + (k + 1) \sum_{|\gamma| = k + 1} \int_0^{|z|} \frac{(|z| - t)^k}{\gamma!} (z')^\gamma D^\gamma K_{\alpha,\ell}(t(z' - y)) dt$$

for $z \in U_x$ where $z' = z/|z|$ ($z \neq 0$) and $0' = 0$. In particular, since $x$ belongs to $U_x$, we have

$$K_{\alpha,\ell;k}(x, y) = (k + 1) \sum_{|\gamma| = k + 1} \int_0^{|x|} \frac{(|x| - t)^k}{\gamma!} (x')^\gamma D^\gamma K_{\alpha,\ell}(tx' - y) dt.$$

We also note that $d(y, \ell_x) > |x|/2$ implies that $|y|/3 < |tx' - y| < 3|y|$ for $0 \leq t \leq |x|$. Therefore by (3.1), for $d(y, \ell_x) > |x|/2$

$$|K_{\alpha,\ell;k}(x, y)| \leq (k + 1) \sum_{|\gamma| = k + 1} \int_0^{|x|} \frac{(|x| - t)^k}{\gamma!} |D^\gamma K_{\alpha,\ell}(tx' - y)| dt$$

$$\leq C(k + 1) \sum_{|\gamma| = k + 1} \int_0^{|x|} \frac{(|x| - t)^k}{\gamma!} |tx' - y|^{\alpha - n - |\gamma|} (1 + |\log |tx' - y||)^{\ell} dt$$

$$\leq C \sum_{|\gamma| = k + 1} |y|^{\alpha - n - |\gamma|} (1 + |\log |y||)^{\ell} \int_0^{|x|} (|x| - t)^k dt$$

$$= C|x|^{k+1} |y|^{\alpha - n - k - 1} (1 + |\log |y||)^{\ell}. $$

Thus we obtain the lemma. □

For a function $f$ we set

$$K_{\alpha,\ell;k} f(x) = \int K_{\alpha,\ell;k}(x, y) f(y) dy.$$

The main purpose of this section is to prove the following integral estimate.
Let \((1/p) + (1/p') = 1\).

**Theorem 3.2** Let \(\alpha > 0, \ p > 1, \ r > -n/p', \ \ell \in \mathbb{N}, \ s \in \mathbb{R} \) and \(\alpha + r - (n/p) \notin \mathbb{N}\). Then for \(k = \lfloor \alpha + r - (n/p) \rfloor\)

\[
\|K_{\alpha,\ell,k}f\|_{p; -\alpha, s - \ell} \leq C\|f\|_{p; -r, s}.
\]

For \(k, \ell \in \mathbb{N} \) and \(r, s \in \mathbb{R}\) we set

\[
K_{\alpha,\ell,k}^{r,s}(x,y) = |x|^{-\alpha-r}(1 + |\log |x||)^{s-\ell}K_{\alpha,\ell,k}(x,y)|y|^r(1 + |\log |y||)^{-s}
\]

and

\[
K_{\alpha,\ell,k}^{r,s}f(x) = \int K_{\alpha,\ell,k}^{r,s}(x,y)f(y)dy.
\]

Obviously, in order to prove Theorem 3.2 it is sufficient to show the following proposition.

**Proposition 3.3** Let \(\alpha > 0, \ p > 1, \ r > -n/p', \ \ell \in \mathbb{N}, \ s \in \mathbb{R} \) and \(\alpha + r - (n/p) \notin \mathbb{N}\). Then for \(k = \lfloor \alpha + r - (n/p) \rfloor\)

\[
\|K_{\alpha,\ell,k}^{r,s}f\|_{p} \leq C\|f\|_{p}.
\]

To show Proposition 3.3 we prepare four lemmas. The first lemma is a special case of the inequality by G. O. Okikiolu [Ok, Theorem 2.1].

**Lemma 3.4** Let \(K(x,y)\) be a nonnegative measurable function on \(\mathbb{R}^n \times \mathbb{R}^n\). Suppose that there are a measurable function \(\varphi(x) > 0\) on \(\mathbb{R}^n\) and constants \(M_1 > 0, \ M_2 > 0\) such that

\[
\int \varphi(y)^pK(x,y)dy \leq M_1^p\varphi(x)^p \quad (3.2)
\]

\[
\int \varphi(x)^pK(x,y)dx \leq M_2^p\varphi(y)^p. \quad (3.3)
\]

If the operator \(K\) is defined by

\[
Kf(x) = \int K(x,y)f(y)dy,
\]
then
\[ \|Kf\|_p \leq M_1 M_2 \|f\|_p. \]

**Lemma 3.5** Let \( \alpha > 0 \), \( \ell \in \mathbb{N} \), \( r > -n/p' \) and \( s \in \mathbb{R} \). Then
\[
\left( \int_{|x-y|\leq 3|x|/2} |x|^{-\alpha-r}(1 + |\log |x||)^{s-\ell}|x-y|^{\alpha-n}(1 + |\log |x-y||)^{\ell}|y|^r \right.
\times (1 + |\log |y||)^{-s}f(y)dy \left| dx \right|^{1/p} \leq C \|f\|_p.
\]

**Proof.** Let
\[
K(x, y) = \begin{cases} 
|x|^{-\alpha-r}(1 + |\log |x||)^{s-\ell}|x-y|^{\alpha-n} \\
	imes (1 + |\log |x-y||)^{\ell}|y|^r(1 + |\log |y||)^{-s}, & |x-y| \leq 3|x|/2 \\
0, & |x-y| > 3|x|/2.
\end{cases}
\]

The condition \( r > -n/p' \) implies that \(-(r+n)/p'\) < \( r/p \), and hence we can take a number \( a \) such that \(-((r+n)/p') < a < r/p \). For the above \( K(x, y) \) and \( \varphi(x) = |x|^a(1 + |\log |x||)^b \) with \( b > \max(s/p', (\ell - s)/p) \) we prove (3.2) and (3.3). First we have
\[
I(x) = \int \varphi(y)^p K(x, y)dy
\]
\[
= \int_{|x-y|\leq 3|x|/2} |y|^{ap'}(1 + |\log |y||)^{bp'}|x|^{-\alpha-r}(1 + |\log |x||)^{s-\ell}|x-y|^{\alpha-n} \\
\times (1 + |\log |x-y||)^{\ell}|y|^r(1 + |\log |y||)^{-s}dy
\]
\[
= \int_{|x'-(y/|x|)|\leq 3/2} |y|^{ap'+r}(1 + |\log |y||)^{bp'-s}|x|^{-\alpha-r}(1 + |\log |x||)^{s-\ell} \\
|x^{\alpha-n}|x' - \frac{y}{|x|} \left| \frac{\alpha-n}{1 + \log \left( |x|/|x'-\frac{y}{|x|}\right) \right|^{\ell} dy.
\]

By putting \( z = y/|x| \), we obtain
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\[ I(x) = \int_{|x' - z| \leq 3/2} |x|^{ap'} |z|^{ap' + r} (1 + |\log(|x||z|)|)^{bp' - s} |x|^{-r - n} \]
\[ \times (1 + |\log |x||)^{s - \ell} |x' - z|^{\alpha - n} (1 + |\log(|x'||z|)|) \ell |x|^{n} dz. \]

Noting that \( 1 + |\log(uv)| \leq (1 + |\log u|)(1 + |\log v|) \) for \( u, v > 0 \) and \( bp' - s > 0 \), we get

\[ I(x) = |x|^{ap'} (1 + |\log |x||)^{bp'} \int_{|x' - z| \leq 3/2} |z|^{ap' + r} (1 + |\log |z||)^{bp' - s} \]
\[ \times |x' - z|^{\alpha - n} (1 + |\log |x' - z||) \ell dz. \]

Since \( \alpha > 0 \) and \( a > -((r + n)/p') \), the integral

\[ \int_{|x' - z| \leq 3/2} |z|^{ap' + r} (1 + |\log |z||)^{bp' - s} |x' - z|^{\alpha - n} (1 + |\log |x' - z||) \ell dz \]
exists and is a constant. Hence

\[ I(x) \leq C |x|^{ap'} (1 + |\log |x||)^{bp'} = C \varphi(x)^{p'}. \]

Next we have

\[ J(y) = \int \varphi(x)^{p} K(x, y) dx \]
\[ = \int_{|x| \geq 2|x - y|/3} |x|^{ap} (1 + |\log |x||)^{bp}|x|^{-\alpha - r} (1 + |\log |x||)^{s - \ell} |x - y|^{\alpha - n} \]
\[ \times (1 + |\log |x - y||)^{\ell} |y|^{r} (1 + |\log |y||)^{-s} dx \]
\[ = \int_{|x/y| \geq 2|x/y| - y'/3} |x|^{ap - \alpha - r} (1 + |\log |x||)^{bp + s - \ell} |y|^{\alpha - n} \left| \frac{x}{|y|} - \frac{y'}{|y|} \right|^{\alpha - n} \]
\[ \times \left( 1 + \left| \log \left( \frac{|y|}{|y|} - \frac{x}{|y|} \right) \right| \right)^{\ell} |y|^{r} (1 + |\log |y||)^{-s} dx. \]

By putting \( w = x/|y| \), we get
\[
J(y) = \int_{|w| \geq 2|w-y'|/3} |y|^{ap-\alpha-r}|w|^{ap-\alpha-r}(1 + |\log(|w||y||)|)^{bp+s-\ell}|y|^{\alpha-n+r}
\times |w - y'|^{\alpha-n}(1 + |\log(|y||w - y'||)|)\ell(1 + |\log |y||)^{-s}|y|^{n} dw.
\]

Noting that \(bp + s - \ell > 0\), we have
\[
J(y) \leq |y|^{ap}(1 + |\log |y||)^{bp} \int_{|w| \geq 2|w-y'|/3} |w|^{ap-r-\alpha}(1 + |\log |w||)^{bp+s-\ell}
\times |w - y'|^{\alpha-n}(1 + |\log |w - y'||)|\ell dw.
\]

Since \(\alpha > 0\) and \(a < r/p\), the integral
\[
\int_{|w| \geq 2|w-y'|/3} |w|^{ap-r-\alpha}(1 + |\log |w||)^{bp+s-\ell}|w - y'|^{\alpha-n}(1 + |\log |w - y'||)\ell dw
\]
exists and is a constant. Hence
\[
J(y) \leq C|y|^{ap}(1 + |\log |y||)^{bp} = C\varphi(y)^{p}.
\]

Thus we obtain (3.2) and (3.3). This proves the lemma by Lemma 3.4. □

**Lemma 3.6**  
Let \(t - (n/p) > 0\) and \(u \in \mathbb{R}\). Then
\[
\left( \int \left( \int_{|y| \leq 2|x|} |x|^{-t}(1 + |\log |x||)^{-u} |y|^{t-n}(1 + |\log |y||)^u f(y) dy \right)^p dx \right)^{1/p} \leq C\|f\|_p.
\]

**Proof.** Let
\[
K(x, y) = \begin{cases} 
|x|^{-t}(1 + |\log |x||)^{-u} |y|^{t-n}(1 + |\log |y||)^u, & |y| \leq 2|x| \\
0, & |y| > 2|x|.
\end{cases}
\]

For the above \(K(x, y)\) and \(\varphi(x) = |x|^{-n/(pp')} (1 + |\log |x||)^b\) with \(b > \max(-u/p', u/p)\) we prove (3.2) and (3.3). First, by \(bp' + u > 0\) we have
$$I(x) = \int \varphi(y)^p K(x, y) dy$$

$$= \int_{|y| \leq 2|x|} |y|^{-n/p} (1 + |\log |y||)^{bp'} |x|^{-t} (1 + |\log |x||)^{-u} |y|^{t-n}$$

$$\times (1 + |\log |y||)^u dy$$

$$= |x|^{-t} (1 + |\log |x||)^{-u} \int_{|y/|x|| \leq 2} |x|^{-t-(n/p)-n} \left| \frac{y}{|x|} \right|^{t-(n/p)-n}$$

$$\times \left( 1 + |\log \left| \frac{y}{|x|} \right| \right)^{bp'+u} \left| \frac{y}{|x|} \right|^{t-(n/p)-n}$$

$$\leq |x|^{-(n/p)-n} (1 + |\log |x||)^{bp'} \int_{|y/|x|| \leq 2} \left| \frac{y}{|x|} \right|^{t-(n/p)-n}$$

$$\times \left( 1 + |\log \left| \frac{y}{|x|} \right| \right)^{bp'+u} \left| \frac{y}{|x|} \right|^{t-(n/p)-n}$$

By putting $z = y/|x|$ we get

$$I(x) \leq |x|^{-n/p} (1 + |\log |x||)^{bp'} \int_{|z| \leq 2} |z|^{t-(n/p)-n} (1 + |\log |z||)^{bp'+u} dz$$

$$= C|x|^{-n/p} (1 + (|\log |x||)^{bp'} = C \varphi(x)^p$$

because of $t - (n/p) > 0$. Next, by $bp - u > 0$ we have

$$J(y) = \int \varphi(x)^p K(x, y) dx$$

$$= \int_{|x| \geq |y|/2} |x|^{-n/p'} (1 + |\log |x||)^{bp} |x|^{-t} (1 + |\log |x||)^{-u} |y|^{t-n}$$

$$\times (1 + |\log |y||)^u dx$$

$$= |y|^{t-n} (1 + |\log |y||)^u \int_{|x/|y|| \geq 1/2} |y|^{t-(n/p')} \left| \frac{x}{|y|} \right|^{t-(n/p')}$$

$$\times \left( 1 + |\log \left( \frac{|y|}{|x|} \right) \right)^{bp-u} dx$$
\[ \leq |y|^{-(n/p')-\eta}(1 + |\log |y||)^{bp} \int_{|x/y| \geq 1/2} \left| \frac{x}{|y|} \right|^{t-(n/p')} \]

\[ \times \left( 1 + |\log \left| \frac{x}{|y|} \right| \right)^{bp-u} dx. \]

By putting \( w = x/|y| \) we obtain

\[ J(y) \leq |y|^{-n/p'}(1 + |\log |y||)^{bp} \int_{|w| \geq 1/2} |w|^{t-(n/p')}(1 + |\log |w||)^{bp-u} dw \]

\[ = C|y|^{-n/p'}(1 + |\log |y||)^{bp} = C\varphi(y)^p \]

because \( t - (n/p) > 0 \) implies \( -t - (n/p') < -n \). Thus we obtain (3.2) and (3.3). Therefore the lemma is proved by Lemma 3.4.

**Lemma 3.7** Let \( t - (n/p) < 0 \) and \( u \in \mathbb{R} \). Then

\[ \left( \int |x|^{-t}(1 + |\log |x||)^{-u} |y|^{t-n} \left( 1 + |\log |y|| \right)^{u} f(y) dy \right)^{1/p} \]

\[ \leq C \|f\|_p. \]

**Proof.** We denote the left-hand side by \( I \). Since the Jacobian of the change of variables \( y = z/|z|^2 \) is 1/|z|^{2n}, by the change of variables and putting \( g(z) = |z|^{-2n/p} f(z/|z|^2) \) we have

\[ I = \left( \int \left| \int_{|z| \leq 2/|x|} |x|^{-t}(1 + |\log |x||)^{-u} |z|^{n-t} \left( 1 + |\log \frac{1}{|z|} \right)^u \right. \]

\[ \times |z|^{2n/p} |z|^{-2n/p} f \left( \frac{z}{|z|^2} \right) \frac{1}{|z|^{2n}} d^p \right)^{1/p} \]

\[ = \left( \int \left| \int_{|z| \leq 2/|x|} |x|^{-t}(1 + |\log |x||)^{-u} |z|^{-n-t+(2n/p)} \right. \]

\[ \times \left. (1 + |\log |z||)^u g(z) d^p \right)^{1/p}. \]

Again by using the change of variables \( x = w/|w|^2 \) we get
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\[ I = \left( \int \int_{|z| \leq 2|w|} |w|^t \left( 1 + \log \frac{1}{|w|} \right)^{-u} |z|^{-n-t+(2n/p)} \right. \]
\[ \times \left( 1 + \log |z| \right) u g(z) \right) \frac{p \ dw}{|w|^{2n}} \right)^{1/p} \]
\[ = \left( \int \int_{|z| \leq 2|w|} |w|^{-t-(2n/p)} \left( 1 + \log |w| \right) - u |z|^{-t+(2n/p)-n} \right. \]
\[ \times \left( 1 + \log |z| \right) u g(z) \right) \frac{p \ dw}{|w|^{2n}} \right)^{1/p}. \]

By putting \( v = -t + (2n/p) \), we see that

\[ I = \left( \int \int_{|z| \leq 2|w|} |w|^{-v} \left( 1 + \log |w| \right) - u |z|^{-v-n \left( 1 + \log |z| \right)} u g(z) \right) \frac{p \ dw}{|w|^{2n}} \right)^{1/p}. \]

Since \( t - (n/p) < 0 \) implies \( v - (n/p) > 0 \), Lemma 3.6 gives \( I \leq C \|g\|_p \). Noting that \( \|g\|_p = \|f\|_p \), we obtain the lemma. \( \square \)

Now we are in a position to prove Proposition 3.3.

**Proof of Proposition 3.3.** We put \( I = \|K_{\alpha, \ell, k}^r f\|_p \). In case of \( \alpha + r - (n/p) > 0 \) we have

\[ I = \left( \int \int_{d(y, \ell) \leq |x|/2} |x|^{-\alpha-r} \left( 1 + \log |x| \right)^{s-\ell} K_{\alpha, \ell, k}(x, y) \right| y \right)^r \]
\[ \times \left( 1 + \log |y| \right)^{-s} f(y) \right) \frac{p \ dy}{|y|^{2n}} \right)^{1/p} \]
\[ + \int_{d(y, \ell) > |x|/2} |x|^{-\alpha-r} \left( 1 + \log |x| \right)^{s-\ell} K_{\alpha, \ell, k}(x, y) \right| y \right)^r \]
\[ \times \left( 1 + \log |y| \right)^{-s} f(y) \right) \frac{p \ dy}{|y|^{2n}} \right)^{1/p} \]
\[ \leq \left( \int \int_{d(y, \ell) \leq |x|/2} |x|^{-\alpha-r} \left( 1 + \log |x| \right)^{s-\ell} \right. \]
\[ \times \left. \left| K_{\alpha, \ell}(x - y) - \sum_{|\gamma| \leq k} \frac{x^\gamma}{\gamma!} D^\gamma K_{\alpha, \ell}(-y) \right| \right) \right)^{1/p} \]

By putting \( v = -t + (2n/p) \), we see that

\[ I = \left( \int \int_{|z| \leq 2|w|} |w|^{-v} \left( 1 + \log |w| \right) - u |z|^{-v-n \left( 1 + \log |z| \right)} u g(z) \right) \frac{p \ dw}{|w|^{2n}} \right)^{1/p}. \]

Since \( t - (n/p) < 0 \) implies \( v - (n/p) > 0 \), Lemma 3.6 gives \( I \leq C \|g\|_p \). Noting that \( \|g\|_p = \|f\|_p \), we obtain the lemma. \( \square \)

Now we are in a position to prove Proposition 3.3.
\[
\times |y|^r (1 + |\log |y||)^{-s} |f(y)| dy \right)^p dx \right)^{1/p} \\
+ \left( \int \left( \int_{d(y, \ell_x) > |x|/2} |x|^{-\alpha - r} (1 + |\log |x||)^{s-\ell} |K_{\alpha, \ell; k}(x, y)||y|^r \\
\times (1 + |\log |y||)^{-s} |f(y)| dy \right)^p dx \right)^{1/p} \\
= I_1 + I_2.
\]

Since \(d(y, \ell_x) \leq |x|/2\) implies that \(|x - y| \leq 3|x|/2\) and \(|y| \leq 3|x|/2\), by (3.1) we see that

\[
I_1 \leq \left( \int \left( \int_{|x - y| \leq 3|x|/2} |x|^{-\alpha - r} (1 + |\log |x||)^{s-\ell} |K_{\alpha, \ell; k}(x - y)||y|^r \\
\times (1 + |\log |y||)^{-s} |f(y)| dy \right)^p dx \right)^{1/p} \\
+ \sum_{|\gamma| \leq k} \frac{1}{\gamma!} \left( \int \left( \int_{|y| \leq 3|x|/2} |x|^{-\alpha - r + |\gamma|} (1 + |\log |x||)^{s-\ell} |D^\gamma K_{\alpha, \ell; k}(-y)| \\
\times |y|^r (1 + |\log |y||)^{-s} |f(y)| dy \right)^p dx \right)^{1/p} \\
\leq C \left( \int \left( \int_{|x - y| \leq 3|x|/2} |x|^{-\alpha - r} (1 + |\log |x||)^{s-\ell} |x - y|^{\alpha - n} \\
\times (1 + |\log |x - y||)^{\ell} |y|^r (1 + |\log |y||)^{-s} |f(y)| dy \right)^p dx \right)^{1/p} \\
+ C \sum_{|\gamma| \leq k} \left( \int \left( \int_{|y| \leq 3|x|/2} |x|^{-(\alpha + r - |\gamma|)} (1 + |\log |x||)^{s-\ell} |y|^{\alpha + r - |\gamma| - n} \\
\times (1 + |\log |y||)^{\ell - s} |f(y)| dy \right)^p dx \right)^{1/p} \\
= I_{11} + I_{12}.
\]

Since \(\alpha > 0, r > -n/p', \ell \in \mathbb{N}\) and \(s \in \mathbb{R}\), Lemma 3.5 gives that \(I_{11} \leq\)
$C\|f\|_p$. The conditions $\alpha + r - (n/p) > 0$ and $\alpha + r - (n/p) \notin \mathbb{N}$ imply that $\alpha + r - |\gamma| - (n/p) > 0$ for $|\gamma| \leq k = [\alpha + r - (n/p)]$. Hence, by Lemma 3.6 we get

\[
I_{12} \leq C \sum_{|\gamma| \leq k} \|f\|_p = C\|f\|_p.
\]

By Lemma 3.1 and the fact that $d(y, \ell x) > |x|/2$ implies $|y| > |x|/2$, we see that

\[
I_2 \leq \left( \int \left( \int_{|y| > |x|/2} |x|^{-(\alpha + r - k - 1)}(1 + |\log |x||)^{s-\ell} |y|^\alpha + r - k - 1 - n \times (1 + |\log |y||)^{\ell-s}|f(y)|dy \right)^p dx \right)^{1/p}.
\]

Since $k = [\alpha + r - (n/p)]$, we have $\alpha + r - k - 1 - (n/p) < 0$, and hence Lemma 3.7 gives $I_2 \leq C\|f\|_p$. Consequently $I \leq C\|f\|_p$. Next we consider the case $\alpha + r - (n/p) < 0$. In this case, since $K_{\alpha, \ell, k}(x, y) = K_{\alpha, \ell}(x - y)$, by (3.1) we have

\[
I = \left( \int \left| \int |x|^{-\alpha-r}(1 + |\log |x||)^{s-\ell} K_{\alpha, \ell}(x - y)\right|^p dy \right)^{1/p}
\]

\[
\leq C \left( \int \left( \int_{d(y, \ell x) \leq |x|/2} |x|^{-\alpha-r}(1 + |\log |x||)^{s-\ell}|x - y|^{\alpha-n} \times (1 + |\log |y||)^{\ell}|y|^r(1 + |\log |y||)^{-s}|f(y)|dy \right)^p dx \right)^{1/p}
\]

\[
+ C \left( \int \left( \int_{d(y, \ell x) > |x|/2} |x|^{-\alpha-r}(1 + |\log |x||)^{s-\ell}|x - y|^{\alpha-n} \times (1 + |\log |x - y||)^{\ell}|y|^r(1 + |\log |y||)^{-s}|f(y)|dy \right)^p dx \right)^{1/p}
\]

\[
= J_1 + J_2.
\]
Since \( d(y, \ell_x) \leq |x|/2 \) implies \(|x-y| \leq 3|x|/2\), the conditions \( \alpha > 0, \ell \in \mathbb{N}, r > -n/p' \) and \( s \in \mathbb{R} \) allows us to apply Lemma 3.5 to \( J_1 \). Then we get \( J_1 \leq C\|f\|_p \). Since \( d(y, \ell_x) > |x|/2 \) implies \(|y| > |x|/2 \) and \(|y|/3 < |x-y| < 3|y|\), the condition \( \alpha + r - (n/p) < 0 \) and Lemma 3.7 gives

\[
J_2 \leq \left( \int \left( \int_{|y|>|x|/2} |x|^{-\alpha-r}(1 + |\log |x||)^{s-\ell} |y|^{|\alpha+r-n|/2} 
\times (1 + |\log |y||)^{\ell-s} |f(y)| dy \right)^p dx \right)^{1/p} 
\leq C\|f\|_p.
\]

Therefore \( I \leq C\|f\|_p \). Thus we complete the proof of Proposition 3.3. \( \square \)

By applying Theorem 3.2 to the Riesz potentials we obtain the following corollary.

**Corollary 3.8** Let \( r > -n/p', s \in \mathbb{R} \) and \( \alpha + r - (n/p) \notin \mathbb{N} \). Then for \( k = [\alpha + r - (n/p)] \)

\[
\begin{cases}
\|U_{\alpha,k}f\|_{p,-\alpha-r,s} \leq C\|f\|_{p,-r,s}, & \alpha - n \notin 2\mathbb{N} \\
\|U_{\alpha,k}f\|_{p,-\alpha-r,s-1} \leq C\|f\|_{p,-r,s}, & \alpha - n \in 2\mathbb{N}.
\end{cases}
\]

4. A semi-group formula for Riesz potentials of \( L^p \)-functions

In Section 2 we stated that for \( \varphi \in \Phi(\mathbb{R}^n), U_{\varphi} \in \Phi(\mathbb{R}^n) \) and hence \( U_\alpha(U_{\alpha+\beta}\varphi) \in \Phi(\mathbb{R}^n) \). Moreover, we referred to the fact that the equality \( U_{\alpha+\beta}\varphi = U_\alpha(U_{\alpha+\beta}\varphi) \) holds for \( \varphi \in \Phi(\mathbb{R}^n) \). Let \( f \in L^p(\mathbb{R}^n) \). We consider the case \( \beta - (n/p) \notin \mathbb{N} \) and \( \alpha + \beta - (n/p) \notin \mathbb{N} \). According to Theorem 3.2 \( U_{\beta,[(\beta-(n/p))^+]f} \) belongs to \( L^{p,-\beta,-1}(\mathbb{R}^n) \). Therefore again by Theorem 3.2 \( U_{\alpha,[(\alpha+\beta-(n/p))^+]f} \) belongs to \( L^{p,-\alpha-\beta,-2}(\mathbb{R}^n) \). On the other hand, it follows also from Theorem 3.2 that \( U_{\alpha+\beta,[(\alpha+\beta-(n/p))^+]f} \) belongs to \( L^{p,-\alpha-\beta,-1}(\mathbb{R}^n) \). The purpose of this section is to prove that the both are equal (a semi-group formula).

We begin with some remarks.

**Remark 4.1** We denote by \( L^1_{loc}(\mathbb{R}^n) \) the space of all locally integrable functions in \( \mathbb{R}^n \). If \( r > -n/p' \), then \( L^{p,-r,s}(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n) \).
Remark 4.2 Let \( f \in L^{p,-r,s}(\mathbb{R}^n) \cap S(\mathbb{R}^n) \). Then the Riesz polynomial \( P_{\alpha,k}f \) of type \( (\alpha, k) \) of \( f \) exists for \( k < \alpha + r - (n/p) \).

Remark 4.3 Let \( r - (n/p) > 0 \), \( r - (n/p) \notin \mathbb{N} \), \( s \in \mathbb{R} \) and \( P(x) \) is a polynomial of degree \( [r - (n/p)] \). If \( P(x) \in L^{p,-r,s}(|x| \leq 1) \), then \( P = 0 \) where

\[
L^{p,-r,s}(|x| \leq 1) = \left\{ f : \int_{|x|\leq 1} |f(x)|^p |x|^{-rp}(1 + |\log |x||)^{sp} dx < \infty \right\}.
\]

Now we prove our main theorem.

Theorem 4.4 Let \( \beta - (n/p) \notin \mathbb{N} \), \( \alpha + \beta - (n/p) \notin \mathbb{N} \) and \( f \in L^p(\mathbb{R}^n) \). Then

\[
U_{\alpha+\beta,[\alpha+\beta-(n/p)]}f = U_{\alpha,[\alpha+\beta-(n/p)]}(U_{\beta,[\beta-(n/p)]}f).
\]

Proof. Let \( f \in L^p(\mathbb{R}^n) \). Since \( \Phi_{\beta,[\beta-(n/p)]}(\mathbb{R}^n) \) is dense in \( L^p(\mathbb{R}^n) \) by Proposition 2.9, there exists a sequence \( \{ \varphi_m \} \subset \Phi_{\beta,[\beta-(n/p)]}(\mathbb{R}^n) \) such that \( \varphi_m \) converges to \( f \) in \( L^p(\mathbb{R}^n) \) as \( m \to \infty \). Since \( \varphi_m \in \Phi(\mathbb{R}^n) \), Proposition 2.1 gives

\[
U_{\alpha+\beta}(\varphi_m) = U_\alpha(U_\beta \varphi_m). \tag{4.1}
\]

Moreover, since \( \varphi_m \in \Phi(\mathbb{R}^n) \subset S(\mathbb{R}^n) \) and \( [\alpha+\beta-(n/p)] < \alpha+\beta \), by (2.1) we have

\[
U_{\alpha+\beta,[\alpha+\beta-(n/p)]} \varphi_m = U_{\alpha+\beta} \varphi_m + P_{\alpha+\beta,[\alpha+\beta-(n/p)]} \varphi_m. \tag{4.2}
\]

On the other hand, the fact \( \varphi_m \in \Phi_{\beta,[\beta-(n/p)]}(\mathbb{R}^n) \) gives \( P_{\beta,[\beta-(n/p)]} \varphi_m = 0 \), and hence \( U_{\beta,[\beta-(n/p)]} \varphi_m = U_\beta \varphi_m \). By using Proposition 2.1 and Theorem 3.2 we see that \( U_{\beta} \varphi_m \in \Phi(\mathbb{R}^n) \cap L^{p,-\beta,-1}(\mathbb{R}^n) \). The fact \( U_\beta \varphi_m \in \Phi(\mathbb{R}^n) \) implies the existence of \( U_\alpha(U_\beta \varphi_m) \), and the fact \( U_\beta \varphi_m \in \Phi(\mathbb{R}^n) \cap L^{p,-\beta,-1}(\mathbb{R}^n) \) gives the existence of \( P_{\alpha,[\alpha+\beta-(n/p)]}(U_\beta \varphi_m) \) by \( [\alpha+\beta-(n/p)] < \alpha+\beta-(n/p) \) and Remark 4.2. Hence

\[
U_{\alpha,[\alpha+\beta-(n/p)]}(U_{\beta,[\beta-(n/p)]} \varphi_m) = U_{\alpha,[\alpha+\beta-(n/p)]}(U_\beta \varphi_m)

= U_\alpha(U_\beta \varphi_m) + P_{\alpha,[\alpha+\beta-(n/p)]}(U_\beta \varphi_m). \tag{4.3}
\]
By (4.1), (4.2) and (4.3) we obtain
\[
U_{\alpha + \beta, \beta - (n/p)} \varphi_m - U_{\alpha, \beta - (n/p)} (U_{\beta, \beta - (n/p)} \varphi_m)
= P_{\alpha + \beta, \beta - (n/p)} (U_{\beta} \varphi_m) - P_{\alpha, \beta - (n/p)} \varphi_m.
\]
(4.4)

Since \(\beta - (n/p) \notin \mathbb{N}\) and \(\alpha + \beta - (n/p) \notin \mathbb{N}\), Theorem 3.2 implies that the left-hand side of (4.4) belongs to \(L^{p, \alpha - \beta, -2}(\mathbb{R}^n)\). Therefore the right-hand side of (4.4) also belongs to \(L^{p, -\alpha - \beta, 2}(\mathbb{R}^n)\), and is a polynomial of degree \(\lfloor \alpha + \beta - (n/p) \rfloor\). This shows that the right-hand side of (4.4) is zero by Remark 4.3. Thus we obtain
\[
U_{\alpha + \beta, \beta - (n/p)} \varphi_m = U_{\alpha, \beta - (n/p)} (U_{\beta, \beta - (n/p)} \varphi_m).
\]
(4.5)

Next we consider the limit process as \(m \to \infty\) in (4.5). Since \(\varphi_m\) converges to \(f\) in \(L^p(\mathbb{R}^n)\) as \(m \to \infty\) and \(\alpha + \beta - (n/p) \notin \mathbb{N}\), by Theorem 3.2 \(U_{\alpha + \beta, \beta - (n/p)} \varphi_m\) converges to \(U_{\alpha + \beta, \beta - (n/p)} f\) in \(L^{p, \alpha - \beta, -1}(\mathbb{R}^n)\), and hence in \(L^1_{1oc}(\mathbb{R}^n)\) as \(m \to \infty\) by \(\alpha + \beta > 0 > -n/p'\) and Remark 4.1. On the other hand, \(U_{\beta, \beta - (n/p)} \varphi_m\) converges to \(U_{\beta, \beta - (n/p)} f\) in \(L^{p, -\beta, -1}(\mathbb{R}^n)\) as \(m \to \infty\) on account of \(\beta - (n/p) \notin \mathbb{N}\) and Theorem 3.2. Hence by using Theorem 3.2 again, we see that \(U_{\alpha, \beta - (n/p)} (U_{\beta, \beta - (n/p)} \varphi_m)\) converges to \(U_{\alpha, \beta - (n/p)} (U_{\beta, \beta - (n/p)} f)\) in \(L^{p, -\alpha - \beta, -2}(\mathbb{R}^n)\), and hence in \(L^1_{1oc}(\mathbb{R}^n)\) as \(m \to \infty\) because of \(\alpha + \beta - (n/p) \notin \mathbb{N}\). This fact and (4.5) implies that
\[
U_{\alpha + \beta, \beta - (n/p)} f = U_{\alpha, \beta - (n/p)} (U_{\beta, \beta - (n/p)} f).
\]

We complete the proof in Theorem 4.4. □

Finally, we give an improvement of the integral estimates in Corollary 3.8 by using the semi-group formula in Theorem 4.4.

**Corollary 4.5** Let \(\alpha - (n/p) \notin \mathbb{N}\) and \(f \in L^p(\mathbb{R}^n)\). Then for \(k = [\alpha - (n/p)]\)
\[
\|U_{\alpha, k} f\|_{p, -\alpha, 0} \leq C \|f\|_p.
\]

**Proof.** In case of \(\alpha - n \notin 2\mathbb{N}\), this is nothing but Corollary 3.8. Let \(\alpha - n \in 2\mathbb{N}\). We can take positive numbers \(\beta\) and \(\zeta\) such that \(\alpha = \beta + \zeta, \beta - n \notin 2\mathbb{N}\), \(\zeta - n \notin 2\mathbb{N}\) and \(\zeta - (n/p) \notin \mathbb{N}\). Since \(\alpha - (n/p) = \beta + \zeta - (n/p) \notin \mathbb{N}\) and
\( \zeta - (n/p) \notin \mathbb{N} \), by using the semi-group formula in Theorem 4.4 we see that
\[
U_{\alpha,k} f = U_{\beta + \zeta, [\beta + \zeta - (n/p)]} f = U_{\beta, [\beta + \zeta - (n/p)]} (U_{\zeta, [\zeta - (n/p)]} f).
\]
Moreover, by \( \beta + \zeta - (n/p) \notin \mathbb{N} \) and \( \beta - n \notin 2\mathbb{N} \) Theorem 3.2 implies that
\[
\| U_{\alpha,k} f \|_{p,-\alpha,0} = \| U_{\beta, [\beta + \zeta - (n/p)]} (U_{\zeta, [\zeta - (n/p)]} f) \|_{p,-\beta-\zeta,0} \\
\leq C \| U_{\zeta, [\zeta - (n/p)]} f \|_{p,-\zeta,0}.
\] (4.6)
Further, since \( \zeta - (n/p) \notin \mathbb{N} \), \( \zeta - n \notin 2\mathbb{N} \), by Theorem 3.2 again we have
\[
\| U_{\zeta, [\zeta - (n/p)]} f \|_{p,-\zeta,0} \leq C \| f \|_{p,0,0} = \| f \|_p.
\] (4.7)
By combining (4.6) and (4.7) we obtain the required estimate. \( \square \)

References


