Boundedness of maximal operators and Sobolev’s theorem
for non-homogeneous central Morrey spaces of variable exponent

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Abstract. Our aim in this paper is to deal with the boundedness of the Hardy-Littlewood maximal operator in non-homogeneous central Morrey spaces of variable exponent. Further, we give Sobolev’s inequality and Trudinger’s exponential integrability for generalized Riesz potentials.

Key words: Maximal operator, non-homogeneous central Morrey spaces of variable exponent, Riesz potentials, Sobolev’s theorem, Sobolev’s inequality, Trudinger’s exponential integrability.

1. Introduction

Let $\mathbb{R}^N$ be the Euclidean space. In [4], Beurling introduced the space $B^p(\mathbb{R}^N)$ to extend Wiener’s ideas [21], [22] which describes the behavior of functions at infinity. Feichtinger [8] gave an equivalent norm on $B^p(\mathbb{R}^N)$, which is a special case of norms in Herz spaces $K_{p,r}^{\alpha}(\mathbb{R}^N)$ introduced by Herz [12]. Precisely speaking, $B^p(\mathbb{R}^N) = K_{p}^{-N/p,\infty}(\mathbb{R}^N)$ (see also [11]). In [10], García-Cuerva studied the boundedness of the maximal operator on the space $B^p(\mathbb{R}^N)$. As an extension of the space $B^p(\mathbb{R}^N)$, García-Cuerva and Herrero [11] introduced the central Morrey spaces $B^{p,\nu}(\mathbb{R}^N)$ (see also [3]). Alvarez, Guzmán-Partida and Lakey [3] obtained the boundedness of a class of singular integrals operators on the central Morrey spaces (see also Komori [13]), which are more singular than Calderón-Zygmund operators and include pseudo-differential operators.

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition. Our first aim in this paper is to introduce the non-homogeneous central Morrey spaces of variable exponent, and study the boundedness of the Hardy-Littlewood maximal operator (see Theorem 3.1), in a way

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different from Almeida and Drihem [2].

In classical Lebesgue spaces, we know Sobolev’s inequality:

\[ \| I_\alpha f \|_{L^{p^\#}(\mathbb{R}^N)} \leq C \| f \|_{L^p(\mathbb{R}^N)} \]

for \( f \in L^p(\mathbb{R}^N) \), \( 0 < \alpha < N \) and \( 1 < p < N/\alpha \), where \( I_\alpha \) is the Riesz kernel of order \( \alpha \) and \( 1/p^\# = 1/p - \alpha/N \) (see, e.g. [1, Theorem 3.1.4]). This result was extended to the central Morrey spaces by Fu, Lin and Lu [9, Proposition 1.1] (see also Matsuoka and Nakai [15]).

To obtain general results, for \( 0 < \alpha < N \) and an integer \( k \), we define the generalized Riesz potential \( I_{\alpha,k} f \) of order \( \alpha \) of a locally integrable function \( f \) on \( \mathbb{R}^N \) by

\[
I_{\alpha,k} f(x) = \int_{\mathbb{R}^N \setminus B(0,1)} \left\{ I_\alpha(x-y) - \sum_{\{\mu: |\mu| \leq k-1\}} \frac{x^\mu}{\mu!} (D^\mu I_\alpha)(-y) \right\} f(y) \, dy,
\]

where \( I_\alpha(x) = |x|^{\alpha-n} \) (see [16], [17]). Remark here that

\[
I_{\alpha,k} f(x) = \int_{\mathbb{R}^N \setminus B(0,1)} I_\alpha(x-y) f(y) \, dy
\]

when \( k \leq 0 \).

In Section 4, when \( p^+ < N/\alpha \) (see Section 2 for the definition of \( p^+ \)), we shall give Sobolev’s inequality for \( I_{\alpha,k} f \) with functions in the nonhomogeneous central Morrey spaces of variable exponent (see Theorem 4.5); for related result, we refer the reader to Fu, Lin and Lu [9, Theorem 2.1].

In the last section, when \( p = N/\alpha \), we treat Trudinger’s exponential integrability for \( I_{\alpha,k} f \) (see Theorem 5.1).

## 2. Preliminaries

Consider a function \( p(\cdot) \) on \( \mathbb{R}^N \) such that

1. \( 1 < p^- := \inf_{x \in \mathbb{R}^N} p(x) \leq \sup_{x \in \mathbb{R}^N} p(x) =: p^+ < \infty \);
2. \( p(\cdot) \) is log-Hölder continuous, namely

\[
|p(x) - p(y)| \leq \frac{c_p}{\log(e + 1/|x-y|)} \quad \text{for } x, y \in \mathbb{R}^N
\]

with a constant \( c_p \geq 0 \);
(P3) $p(\cdot)$ is log-Hölder continuous at $\infty$, namely

$$|p(x) - p(\infty)| \leq \frac{c_\infty}{\log(e + |x|)}$$

whenever $|x| > 0$

with constants $p(\infty) > 1$ and $c_\infty \geq 0$;

$p(\cdot)$ is referred to as a variable exponent.

For $\nu \geq 0$, we denote by $B_{p(\cdot),\nu}(\mathbb{R}^N)$ the class of locally integrable functions $f$ on $\mathbb{R}^N$ satisfying

$$\|f\|_{B_{p(\cdot),\nu}(\mathbb{R}^N)} = \sup_{R \geq 1} R^{-\nu/p(\infty)} \|f\|_{L_{p(\cdot)}(B(0,R))} < \infty,$$

where

$$\|f\|_{L_{p(\cdot)}(B(0,R))} = \inf \left\{ \lambda > 0 : \int_{B(0,R)} \left( \frac{|f(y)|}{\lambda} \right)^{p(y)} dy \leq 1 \right\}.$$

The space $B_{p(\cdot),\nu}(\mathbb{R}^N)$ is referred to as a non-homogeneous central Morrey spaces of variable exponent. If $p(\cdot)$ is a constant and $\nu = N$, then $B_{p(\cdot),\nu}(\mathbb{R}^N) = B^p(\mathbb{R}^N)$.

Throughout this paper, let $C$ denote various constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1} h \leq g \leq Ch$ for some constant $C > 0$.

**Lemma 2.1** Set

$$\|f\|_{\tilde{B}_{p(\cdot),\nu}(\mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \sup_{R \geq 1} R^{-\nu} \int_{B(0,R)} \left( \frac{|f(y)|}{\lambda} \right)^{p(y)} dy \leq 1 \right\}.$$

Then

$$\|f\|_{B_{p(\cdot),\nu}(\mathbb{R}^N)} \sim \|f\|_{\tilde{B}_{p(\cdot),\nu}(\mathbb{R}^N)}$$

for all $f \in L^1_{loc}(\mathbb{R}^N)$.

**Proof.** We may assume that $\nu > 0$. First we find a constant $C > 0$ such that

$$\|f\|_{B_{p(\cdot),\nu}(\mathbb{R}^N)} \leq C \|f\|_{\tilde{B}_{p(\cdot),\nu}(\mathbb{R}^N)}$$
for all $f \in L^1_{\text{loc}}(\mathbb{R}^N)$. Let $f$ be a nonnegative function on $\mathbb{R}^N$ with $\|f\|_{\tilde{B}^{p(\cdot),\nu}(\mathbb{R}^N)} \leq 1$. Then note that

$$R^{-\nu} \int_{B(0,R)} f(y)^{p(y)} \, dy \leq 1$$

for all $R \geq 1$. To end the proof, it is sufficient to find a constant $C > 0$ such that

$$\int_{B(0,R)\setminus B(0,1)} (R^{-\nu/p(\infty)} f(y))^{p(y)} \, dy \leq C$$

for all $R \geq 1$. For this purpose, let $R \geq 1$ and take an integer $j_0 \geq 1$ such that $2^{-j_0} R \leq 1 < 2^{-j_0+1} R$. We have

$$\int_{B(0,R)\setminus B(0,1)} (R^{-\nu/p(\infty)} f(y))^{p(y)} \, dy$$

$$\leq \sum_{j=0}^{j_0} \int_{B(0,2^{-j+1} R)\setminus B(0,2^{-j} R)} (R^{-\nu/p(\infty)} f(y))^{p(y)} \, dy$$

$$\leq \sum_{j=0}^{j_0} (2^{-j})^{\nu/p(\infty)} \int_{B(0,2^{-j+1} R)\setminus B(0,2^{-j} R)} \left\{ (2^{-j} R)^{-\nu/p(\infty)} f(y) \right\}^{p(y)} \, dy$$

$$\leq \sum_{j=0}^{j_0} (2^{-j})^{\nu/p(\infty)} (2^{-j} R)^{\nu} \int_{B(0,2^{-j+1} R)} f(y)^{p(y)} \, dy$$

$$\leq C \sum_{j=0}^{j_0} (2^{-j})^{\nu/p(\infty)} \leq C$$

since $|y|^{-p(y)} \leq C|y|^{-p(\infty)}$ for $y \in B(0,2^{-j+1} R)\setminus B(0,2^{-j} R)$ and $0 \leq j \leq j_0$ by (P3).

Next we prove the converse inequality. Then it is sufficient to find a constant $C > 0$ such that

$$R^{-\nu} \int_{B(0,R)\setminus B(0,1)} f(y)^{p(y)} \, dy \leq C$$
for all $R \geq 1$ and $f \geq 0$ on $\mathbb{R}^N$ with
\[
\sup_{R > 1} \int_{B(0,R)} \left( R^{-\nu/p(\infty)} f(y) \right)^{p(y)} dy \leq 1.
\]

For this purpose, let $R > 1$ and take an integer $j_0 \geq 1$ such that $2^{-j_0} R \leq 1 < 2^{-j_0+1} R$ as before. We find
\[
\int_{B(0,R) \setminus B(0,1)} \left( R^{-\nu/p(y)} f(y) \right)^{p(y)} dy
\leq \sum_{j=0}^{j_0} (2^{-j})^\nu \int_{B(0,2^{-j+1}R) \setminus B(0,2^{-j}R)} \left\{ (2^{-j} R)^{-\nu/p(y)} f(y) \right\}^{p(y)} dy
\leq \sum_{j=0}^{j_0} (2^{-j})^\nu \int_{B(0,2^{-j+1}R)} \left\{ (2^{-j} R)^{-\nu/p(\infty)} f(y) \right\}^{p(y)} dy
\leq C \sum_{j=0}^{j_0} (2^{-j})^\nu \leq C
\]
since $|y|^{-1/p(y)} \leq C |y|^{-1/p(\infty)}$ for $y \in B(0,2^{-j+1}R) \setminus B(0,2^{-j}R)$ and $0 \leq j \leq j_0$ by (P3). Thus the proof is completed. \hfill \Box

3. **Boundedness of maximal operators**

For a locally integrable function $f$ on $\mathbb{R}^N$, the Hardy-Littlewood maximal function $Mf$ is defined by
\[
Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy,
\]
where $B(x,r)$ is the ball in $\mathbb{R}^N$ with center $x$ and of radius $r > 0$, and $|B(x,r)|$ denotes its Lebesgue measure. The mapping $f \mapsto Mf$ is called the maximal operator.

The maximal operator is a classical tool in harmonic analysis and studying Sobolev functions and partial differential equations, and it plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see [5], [13], [14], [20], etc.).
It is well known that the maximal operator is bounded in the Lebesgue space $L^p(\mathbb{R}^N)$ when $p > 1$ (see [20]). We present the boundedness of maximal operator in the central Morrey spaces of variable exponent.

**Theorem 3.1** Let $0 \leq \nu \leq N$. Then the maximal operator: $f \to Mf$ is bounded from $B^{p(\cdot),\nu}(\mathbb{R}^N)$ to $B^{p(\cdot),\nu}(\mathbb{R}^N)$, that is,

$$
\|Mf\|_{B^{p(\cdot),\nu}(\mathbb{R}^N)} \leq C \|f\|_{B^{p(\cdot),\nu}(\mathbb{R}^N)} \quad \text{for all } f \in B^{p(\cdot),\nu}(\mathbb{R}^N).
$$

When $0 \leq \nu < N$, this theorem is essentially proved by Almeida and Drihem [2, Corollary 4.7]. But, for the readers’ convenience, we give a proof of Theorem 3.1 different from [2].

Before doing this, we prepare the following results.

**Lemma 3.2** ([7, Corollary 4.5.9]) For all $R \geq 1$,

$$
\|1\|_{L^p(B(0,R))} \sim R^{N/p(\infty)},
$$

that is, $1 \in B^{p(\cdot),N}(\mathbb{R}^N)$.

**Lemma 3.3** There exists a constant $C > 0$ such that

$$
\frac{1}{|B(0,R)|} \int_{B(0,R) \setminus B(0,R/2)} f(y) \, dy \leq CR^{-(N-\nu)/p(\infty)}
$$

for all $R \geq 1$ and $f \geq 0$ such that $\|f\|_{B^{p(\cdot),\nu}(\mathbb{R}^N)} \leq 1$.

**Proof.** Let $f$ be a nonnegative function on $\mathbb{R}^N$ such that $\|f\|_{B^{p(\cdot),\nu}(\mathbb{R}^N)} \leq 1$. Then we see from Lemma 2.1 that

$$
R^{-\nu} \int_{B(0,R) \setminus B(0,R/2)} f(y)^{p(y)} \, dy \leq C
$$

for all $R \geq 1$. Hence we find by (P3)

$$
\frac{1}{|B(0,R)|} \int_{B(0,R) \setminus B(0,R/2)} f(y) \, dy \\
\leq R^{-(N-\nu)/p(\infty)} + \frac{1}{|B(0,R)|} \int_{B(0,R) \setminus B(0,R/2)} f(y) \left( \frac{f(y)}{R^{-(N-\nu)/p(\infty)}} \right)^{p(y)-1} \, dy
$$
for all $R \geq 1$, as required.

We denote by $\chi_E$ the characteristic function of $E$.

**Lemma 3.4** Let $0 \leq \nu \leq N$. Then there exists a constant $C > 0$ such that

$$M(f \chi_{\mathbb{R}^N \setminus B(0,2R)})(x) \leq CR^{-(N-\nu)/p(\infty)}$$

for all $x \in B(0,R)$ with $R \geq 1$ and $f \geq 0$ with $\|f\|_{B^p(\cdot,\nu)(\mathbb{R}^N)} \leq 1$.

**Proof.** Let $f$ be a nonnegative function on $\mathbb{R}^N$ such that $\|f\|_{B^p(\cdot,\nu)(\mathbb{R}^N)} \leq 1$. Let $R \geq 1$ and $x \in B(0,R)$. We have by Lemma 3.3

$$M(f \chi_{\mathbb{R}^N \setminus B(0,2R)})(x) = \sup_{r > R} \frac{1}{|B(x,r)|} \int_{B(x,r) \setminus B(0,2R)} f(y) \, dy \leq \sup_{r > R} \frac{1}{|B(0,r)|} \sum_{\{j \geq 1: 2^j R < 2r\}} \int_{B(0,2^{j+1}R) \setminus B(0,2^j R)} f(y) \, dy \leq C \sup_{r > R} \frac{1}{|B(0,r)|} \sum_{\{j \geq 1: 2^j R < 2r\}} (2^j+1)^{N-(N-\nu)/p(\infty)} \leq C \sup_{r > R} \frac{1}{|B(0,r)|} r^{N-(N-\nu)/p(\infty)} \leq CR^{-(N-\nu)/p(\infty)},$$

as required. \qed

We know the following result.

**Lemma 3.5** ([6, Theorem 1.5]) There exists a constant $c_0 > 0$ such that

$$\|Mf\|_{L^p(\mathbb{R}^N)} \leq c_0 \|f\|_{L^p(\mathbb{R}^N)}$$
for all \( f \in L^p(\mathbb{R}^N) \).

Now we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Let \( f \) be a nonnegative function on \( \mathbb{R}^N \) such that \( \|f\|_{B^p\cdot\nu(\mathbb{R}^N)} \leq 1 \). For \( R \geq 1 \), set

\[
f = f \chi_{B(0,2R)} + f \chi_{\mathbb{R}^N \setminus B(0,2R)} = f_1 + f_2.
\]

First we find from Lemmas 3.2 and 3.4

\[
\|Mf_2\|_{L^p(\mathbb{R}^N)} \leq CR^{-(N-\nu)/p(\infty)} \|1\|_{L^p(\mathbb{R}^N)} \leq CR^{-(N-\nu)/p(\infty)} R^{N/p(\infty)} = CR^{\nu/p(\infty)}.
\]

Next we obtain by Lemma 3.5

\[
\|Mf\|_{L^p(\mathbb{R}^N)} \leq \|Mf_1\|_{L^p(\mathbb{R}^N)} + \|Mf_2\|_{L^p(\mathbb{R}^N)} \leq C\{\|f\|_{L^p(\mathbb{R}^N)} + R^{\nu/p(\infty)}\} \leq C\{(2R)^{\nu/p(\infty)} + R^{\nu/p(\infty)}\} \leq CR^{\nu/p(\infty)},
\]

so that

\[
\sup_{R \geq 1} R^{-\nu/p(\infty)}\|Mf\|_{L^p(\mathbb{R}^N)} \leq C.
\]

Thus we establish the required result.

**Remark 3.6** If \( \nu > N \), then, as in the proof of Theorem 3.1, we find

\[
\sup_{R \geq 1} R^{-\nu/p(\infty)}\|M(f \chi_{B(0,R)})\|_{L^p(\mathbb{R}^N)} \leq C.
\]

4. Sobolev’s inequality

For \( \nu \geq 0 \), take the integer \( k \geq 0 \) such that

\[
k - 1 \leq \alpha - (N - \nu)/p(\infty) < k
\]

and consider the generalized Riesz potential
\[ I_{\alpha,k} f(x) = \int_{\mathbb{R}^N \setminus B(0,1)} I_\alpha(x-y) - \sum_{\{\mu : |\mu| \leq k-1\}} \frac{x^\mu}{\mu!} (D^\mu I_\alpha)(-y) f(y) \, dy \]

for a locally integrable function \( f \) on \( \mathbb{R}^N \).

The following estimates are fundamental (see [17] and [19]).

**Lemma 4.1** Let \( k \geq 1 \) be an integer.

1. If \( 2|x| < |y| \), then
   \[
   \left| I_\alpha(x-y) - \sum_{\{\mu : |\mu| \leq k-1\}} \frac{x^\mu}{\mu!} (D^\mu I_\alpha)(-y) \right| \leq C|x|^k |y|^\alpha - N - k;
   \]

2. If \( |x|/2 \leq |y| \leq 2|x| \), then
   \[
   \left| I_\alpha(x-y) - \sum_{\{\mu : |\mu| \leq k-1\}} \frac{x^\mu}{\mu!} (D^\mu I_\alpha)(-y) \right| \leq C|x-y|^{\alpha - N};
   \]

3. If \( 1 \leq |y| \leq |x|/2 \), then
   \[
   \left| I_\alpha(x-y) - \sum_{\{\mu : |\mu| \leq k-1\}} \frac{x^\mu}{\mu!} (D^\mu I_\alpha)(-y) \right| \leq C|x|^{k-1} |y|^{\alpha - N - (k-1)}.
   \]

**Lemma 4.2** Let \( k \) be the integer defined by (4.1). Then there exists a constant \( C > 0 \) such that

\[
|I_{\alpha,k}(f \chi_{\mathbb{R}^N \setminus B(0,2R)})(x)| \leq C R^{\alpha-(N-\nu)/p(\infty)}
\]

for all \( x \in B(0, R) \) with \( R \geq 1 \) and \( f \geq 0 \) with \( \|f\|_{\mathbb{B}^{p(\cdot)}(\nu)(\mathbb{R}^N)} \leq 1 \).

**Proof.** Let \( f \) be a nonnegative function on \( \mathbb{R}^N \) such that \( \|f\|_{\mathbb{B}^{p(\cdot)}(\nu)(\mathbb{R}^N)} \leq 1 \). Let \( R \geq 1 \) and \( x \in B(0, R) \). First note from Lemma 4.1 (1) that

\[
|I_{\alpha,k}(f \chi_{\mathbb{R}^N \setminus B(0,2R)})(x)| \leq C R^k \int_{\mathbb{R}^N \setminus B(0,2R)} |y|^{\alpha-N-k} f(y) \, dy.
\]

Hence, we have by Lemma 3.3
\[
|I_{\alpha,k}(f_{\mathbb{R}^N\setminus B(0,2R)})(x)|
\leq CR^k \sum_{j=1}^{\infty} \int_{B(0,2^{j+1}R)\setminus B(0,2^jR)} |y|^{\alpha-N-k} f(y) \, dy
\leq CR^k \sum_{j=1}^{\infty} (2^j R)^{\alpha-k} \frac{1}{|B(0,2^{j+1}R)|} \int_{B(0,2^{j+1}R)\setminus B(0,2^jR)} f(y) \, dy
\leq CR^k \sum_{j=1}^{\infty} (2^j R)^{\alpha-k-(N-\nu)/p(\infty)}
= CR^{\alpha-(N-\nu)/p(\infty)} \sum_{j=1}^{\infty} 2^j \{\alpha-k-(N-\nu)/p(\infty)\}
\leq CR^{\alpha-(N-\nu)/p(\infty)},
\]

as required. \[\square\]

**Lemma 4.3** Let \( k \geq 1 \) be an integer. Then there exists a constant \( C > 0 \) such that

1. in case \( k-1 < \alpha-(N-\nu)/p(\infty) < k \),

\[
|x|^{k-1} \int_{B(0,|x|/2)\setminus B(0,1)} |y|^{\alpha-N-(k-1)} f(y) \, dy \leq CR^{\alpha-(N-\nu)/p(\infty)};
\]

2. in case \( k-1 = \alpha-(N-\nu)/p(\infty) \),

\[
|x|^{k-1} \int_{B(0,|x|/2)\setminus B(0,1)} |y|^{\alpha-N-(k-1)} f(y) \, dy \leq CR^{\alpha-(N-\nu)/p(\infty)} \log R
\]

for all \( x \in B(0, R) \) with \( R \geq 2 \) and \( f \geq 0 \) with \( \|f\|_{B^p(\cdot,\nu)(\mathbb{R}^N)} \leq 1 \).

**Proof.** Let \( f \) be a nonnegative function on \( \mathbb{R}^N \) such that \( \|f\|_{B^p(\cdot,\nu)(\mathbb{R}^N)} \leq 1 \). Let \( R \geq 2, k \geq 1 \) and \( x \in B(0, R) \). We may assume that \( |x| \geq 2 \). We take an integer \( j_0 \geq 1 \) such that \( 2^{-j_0-1}|x| < 1 \leq 2^{-j_0}|x| \).

First we show the case \( k-1 < \alpha-(N-\nu)/p(\infty) < k \). Then we have by Lemma 3.3
\[ |x|^{k-1} \int_{B(0,|x|/2) \setminus B(0,1)} |y|^{\alpha - N - (k-1)} f(y) \, dy \]
\[ \leq |x|^{k-1} \sum_{j=1}^{j_0} \int_{B(0,2^{-j}|x|) \setminus B(0,2^{-j-1}|x|)} |y|^{\alpha - N - (k-1)} f(y) \, dy \]
\[ \leq C |x|^{k-1} \sum_{j=1}^{\infty} (2^{-j}|x|)^{\alpha - (k-1)} \frac{1}{|B(0,2^{-j}|x|)|} \int_{B(0,2^{-j}|x|) \setminus B(0,2^{-j-1}|x|)} f(y) \, dy \]
\[ \leq CR^{k-1} \sum_{j=1}^{j_0} (2^{-j}R)^{\alpha - (k-1)-(N-\nu)/p(\infty)} \]
\[ \leq CR^{\alpha -(N-\nu)/p(\infty)}. \]

Next we deal with the case \( k - 1 = \alpha - (N - \nu)/p(\infty) \). Since \( j_0 \leq \log |x|/\log 2 < j_0 + 1 \), we see from Lemma 3.3 that
\[ |x|^{k-1} \int_{B(0,|x|/2) \setminus B(0,1)} |y|^{\alpha - N - (k-1)} f(y) \, dy \]
\[ \leq CR^{k-1} \sum_{j=1}^{j_0} (2^{-j}R)^{\alpha - (k-1)-(N-\nu)/p(\infty)} \]
\[ \leq CR^{\alpha -(N-\nu)/p(\infty)} j_0 \]
\[ \leq CR^{\alpha -(N-\nu)/p(\infty)} \log R, \]
as required. \( \square \)

Set
\[ 1/p^*(x) = 1/p(x) - \alpha/N. \]

**Lemma 4.4** ([18, Theorem 4.1]) Suppose \( 1/p^+ - \alpha/N > 0 \). Then there exists a constant \( c_1 > 0 \) such that
\[ \|I_\alpha f\|_{L^{p^*}(\mathbb{R}^N)} \leq c_1 \|f\|_{L^p(\mathbb{R}^N)} \]
for all \( f \in L^p(\mathbb{R}^N) \) with compact support.
Now we show the Sobolev type inequality for generalized Riesz potentials in the central Morrey spaces of variable exponents, as an extension of Fu, Lin and Lu [9] in the constant exponent case.

**Theorem 4.5** (cf. [9, Proposition 1.1]) Suppose \(1/p^+ - \alpha/N > 0\) and \(k - 1 < \alpha - (N - \nu)/p(\infty) < k\). Then there exists a constant \(C > 0\) such that

\[
\sup_{R \geq 1} R^{-\nu/p(\infty)} \|I_{\alpha,k}f\|_{L^{p^*}(\cdot)(B(0,R))} \leq C
\]

for all \(f \geq 0\) with \(\|f\|_{B^{p(\cdot),\nu}(\mathbb{R}^N)} \leq 1\).

**Proof.** Let \(f\) be a nonnegative function on \(\mathbb{R}^N\) such that \(\|f\|_{B^{p(\cdot),\nu}(\mathbb{R}^N)} \leq 1\). For \(R \geq 1\), set

\[
f = f\chi_{B(0,2R)} + f\chi_{\mathbb{R}^N \setminus B(0,2R)} = f_1 + f_2.
\]

First we find by Lemmas 3.2 and 4.2

\[
\|I_{\alpha,k}f_2\|_{L^{p^*}(\cdot)(B(0,R))} \leq CR^{\alpha - (N-\nu)/p(\infty)} \|1\|_{L^{p^*}(\cdot)(B(0,R))} \leq CR^{\alpha - (N-\nu)/p(\infty)} R^N/p^*(\infty) = CR^{\nu/p(\infty)}.
\]

Next, we see from Lemmas 4.1 and 4.3 (1) that

\[
|I_{\alpha,k}f_1(x)| \leq |I_{\alpha,k}(f\chi_{B(0,2R)} \setminus B(0,2|x|))(x)| + |I_{\alpha,k}(f\chi_{B(0,|x|/2)} \setminus B(0,1))(x)|
\]

\[
\leq C\left\{I_{\alpha}f_1(x) + R^{\alpha - (N-\nu)/p(\infty)} \right\}
\]

for \(x \in B(0, R)\) since \(|x|^k|y|^{\alpha - N - k} \leq C|x - y|^{\alpha - N}\) for \(2|x| < |y|\), so that we have by Lemmas 3.2 and 4.4

\[
\|I_{\alpha,k}f\|_{L^{p^*}(\cdot)(B(0,R))} \leq \|I_{\alpha,k}f_1\|_{L^{p^*}(\cdot)(B(0,R))} + \|I_{\alpha,k}f_2\|_{L^{p^*}(\cdot)(B(0,R))} \leq C\left\{\|f\|_{L^{p^*}(\cdot)(B(0,2R))} + R^{\nu/p(\infty)} \right\} \leq CR^{\nu/p(\infty)},
\]
so that
\[
\sup_{R \geq 1} R^{-\nu/p(\infty)} \| I_{\alpha,k} f \|_{L^{p^*(\cdot)}(B(0,R))} \leq C.
\]
Thus we completes the proof. \(\square\)

**Remark 4.6** Suppose \(1/p^+ - \alpha/N > 0\) and \(k - 1 = \alpha - (N - \nu)/p(\infty)\).
Then there exists a constant \(C > 0\) such that
\[
\sup_{R \geq 2} R^{-\nu/p(\infty)} (\log R)^{-1} \| I_{\alpha,k} f \|_{L^{p^*(\cdot)}(B(0,R))} \leq C
\]
for all \(f \geq 0\) with \(\| f \|_{B^{p(\cdot),\nu}(\mathbb{R}^N)} \leq 1\).

5. **Exponential integrability**

Our aim in this section is to discuss the exponential integrability.

**Theorem 5.1** Let \(p = N/\alpha\) and \(k - 1 < \alpha - (N - \nu)/p < k\). Then there exist constants \(c_1, c_2 > 0\) such that
\[
\sup_{R \geq 1} R^{-N} \int_{B(0,R)} \exp \left( \{c_1 R^{-\nu/p} |I_{\alpha,k} f(x)|\}^{p'} \right) dx \leq c_2
\]
for all \(f \geq 0\) with \(\| f \|_{B^{p,\nu}(\mathbb{R}^N)} \leq 1\).

**Proof.** Let \(f\) be a nonnegative function on \(\mathbb{R}^N\) such that \(\| f \|_{B^{p,\nu}(\mathbb{R}^N)} \leq 1\) and let \(x \in B(0,R)\). For \(R \geq 1\), set
\[
f = f \chi_{B(0,2R)} + f \chi_{\mathbb{R}^N \setminus B(0,2R)} = f_1 + f_2.
\]
For \(0 < \delta \leq R\), write
\[
I_\alpha f_1(x) = \int_{B(x,\delta)} |x - y|^{\alpha - N} f(y) dy + \int_{B(0,2R) \setminus B(x,\delta)} |x - y|^{\alpha - N} f(y) dy = U_1(x) + U_2(x).
\]
First we find
\[
U_1(x) \leq C\delta^{\alpha} M f_1(x).
\]
Next we have by Hölder’s inequality
\[ U_2(x) \leq C (\log(2R/\delta))^{1/p'} \|f_1\|_{L^p(B(0,2R))}, \]
so that
\[ I_\alpha f_1(x) \leq C \{ \delta^\alpha Mf_1(x) + (\log(2R/\delta))^{1/p'} R^{\nu/p} \}. \]

Here, letting \( \delta/(2R) = \{ R^{-\nu/p+\alpha} Mf_1(x) \}^{-1/\alpha} (\log(R^{-\nu/p+\alpha} Mf_1(x)))^{1/(\alpha p')} < 1 \), we establish
\[ I_\alpha f_1(x) \leq C (\log(R^{-\nu/p+\alpha} Mf_1(x)))^{1/p'} R^{\nu/p}; \]
if \( \{ R^{-\nu/p+\alpha} Mf_1(x) \}^{-1/\alpha} (\log(R^{-\nu/p+\alpha} Mf_1(x)))^{1/(\alpha p')} \geq 1 \), then, letting \( \delta = R \), we have
\[ I_\alpha f_1(x) \leq CR^{\nu/p}. \]

As in the proof of Theorem 4.5, we see from Lemmas 4.1 and 4.3 (1) that
\[ |I_{\alpha,k}f_1(x)| \leq C \{ I_\alpha f_1(x) + R^{\alpha-(N-\nu)/p} \} = C \{ I_\alpha f_1(x) + R^{\nu/p} \} \]
for \( x \in B(0,R) \), since \( \alpha = N/p. \) Therefore, we obtain
\[ |I_{\alpha,k}f_1(x)| \leq C \{ (\log(e + R^{-\nu/p+\alpha} Mf_1(x)))^{1/p'} R^{\nu/p} + R^{\nu/p} \}. \]

On the other hand, we obtain by Lemma 4.2
\[ |I_{\alpha,k}f_2(x)| \leq CR^{\alpha-(N-\nu)/p} = CR^{\nu/p}, \]
since \( \alpha = N/p. \) Hence, we find
\[ \{ c_1 R^{-\nu/p} |I_{\alpha,k}f(x)| \}^{p'} \leq \log(e + R^{(N-\nu)/p} Mf_1(x)), \]
so that we have by boundedness of maximal operators on \( L^p(\mathbb{R}^N) \)
\[ \int_{B(0,R)} \exp(\{ c_1 R^{-\nu/p} |I_{\alpha,k}f(x)| \}^{p'}) \, dx \leq C \int_{B(0,R)} [1 + R^{N-\nu} \{ Mf_1(x) \}^p] \, dx \]
\[ \leq C \left( R^N + R^{N-\nu} \int_{\mathbb{R}^N} f_1(y)^p \, dy \right) \]
\[ \leq CR^N, \]
as required. \qed

**Remark 5.2** Let \( p = N/\alpha \) and \( k - 1 = \alpha - (N - \nu)/p \). Then there exist constants \( c_1, c_2 > 0 \) such that

\[ \sup_{R \geq 2} R^{-N} \int_{B(0,R)} \exp \left( \{c_1 R^{-\nu/p} (\log R)^{-1} |I_{\alpha,k} f(x)|\}^{p'} \right) \, dx \leq c_2 \]
for all \( f \geq 0 \) with \( \|f\|_{\mathcal{B}^{p,\nu}(\mathbb{R}^N)} \leq 1 \).

**Remark 5.3** If \( p^- \geq p(\infty) \), then \( \mathcal{B}^{p(\cdot),\nu}(\mathbb{R}^N) \subset \mathcal{B}^{p(\infty),\nu}(\mathbb{R}^N) \), and moreover

\[ \|f\|_{\mathcal{B}^{p(\infty),\nu}(\mathbb{R}^N)} \leq C \|f\|_{\mathcal{B}^{p(\cdot),\nu}(\mathbb{R}^N)}. \]

In fact, for \( R \geq 1 \) and \( a > N/p(\infty) \),

\[ R^{-\nu} \int_{B(0,R)} |f(x)|^{p(\infty)} \, dx \]
\[ = R^{-\nu} \int_{\{x \in B(0,R) : |f(x)| \geq 1\}} |f(x)|^{p(\infty)} \, dx \]
\[ + R^{-\nu} \int_{\{x \in B(0,R) : (1+|x|)^{-a} < |f(x)| \leq 1\}} |f(x)|^{p(\infty)} \, dx \]
\[ + R^{-\nu} \int_{\{x \in B(0,R) : |f(x)| \leq (1+|x|)^{-a}\}} |f(x)|^{p(\infty)} \, dx \]
\[ \leq R^{-\nu} \int_{\{x \in B(0,R) : |f(x)| \geq 1\}} |f(x)|^{p(x)} \, dx \]
\[ + R^{-\nu} \int_{\{x \in B(0,R) : (1+|x|)^{-a} < |f(x)| \leq 1\}} |f(x)|^{p(x)} |f(x)|^{p(\infty) - p(x)} \, dx \]
\[ + CR^{-\nu} \int_{B(0,R)} (1 + |x|)^{-ap(\infty)} \, dx \]
\[ \leq C \left\{ R^{-\nu} \int_{B(0,R)} |f(x)|^{p(x)} \, dx + R^{-\nu} \right\} \]

\[ \leq C \]

when \( \|f\|_{B^{p(\cdot),p}(\mathbb{R}^N)} \leq 1. \)

References


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