Bi-flows on a network

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(Received March 6, 2013)

Abstract. Flows on a network play an important role in the theory of discrete harmonic functions. In the study of discrete bi-harmonic functions, we encounter a concept of bi-flows. In this paper, we are concerned with minimization problems for bi-flows which are analogous to those for flows.

Key words: discrete potential theory, bi-harmonic Green function, bi-flows on a network.

1. Introduction

In the theory of discrete potential theory on networks, it is well-known that flows have played an important role related to discrete harmonic functions. For example, a minimizing problem related to flows from a node to the ideal boundary with unit strength characterizes the harmonic Green function. In this paper, we introduce an arc-arc incidence matrix $b(y, y')$ of two arcs $y$ and $y'$ and an operator $B_r$ related to it. We say that a function $w$ on arcs is a bi-flow if $B_r w$ is a flow. If $u$ is a bi-harmonic function defined on nodes, then we see that the discrete derivative $w = du$ is a bi-flow. We shall consider two minimizing problems related to bi-flows from a node to the ideal boundary. The optimal solution of each minimizing problem characterizes the bi-harmonic Green function.

We organize this paper as follows: Some properties of $b$ and $B_r$ will be given in Section 3. We define bi-flows as well as weak bi-flows in Section 4. Two minimizing problems related to bi-flows are given in Sections 5 and 6.

2. Preliminaries

Let $N = \{X, Y, K, r\}$ be an infinite network which is connected and locally finite and has no self-loops. Here $X$ is the set of nodes and $Y$ is the set of arcs. The node-arc incidence matrix $K$ is a function on $X \times Y$ and $K(x, y) = -1$ if $x$ is the initial node $x^-(y)$ of $y$; $K(x, y) = 1$ if $x$
is the terminal node $x^+(y)$ of $y$; otherwise $K(x, y) = 0$. The resistance $r$ is a strictly positive function on $Y$. Let $L(X)$ be the set of all real valued functions on $X$ and let $L_0(X)$ be the set of all $u \in L(X)$ with finite supports. We define $L(Y)$ and $L_0(Y)$ similarly.

For $u \in L(X)$ and $w \in L(Y)$, we define $du \in L(Y)$ and $\partial w \in L(X)$ by

$$du(y) = -r(y)^{-1} \sum_{x \in X} K(x, y)u(x),$$
$$\partial w(x) = \sum_{y \in Y} K(x, y)w(y).$$

Also we define the Laplacian $\Delta u \in L(X)$ and the bi-Laplacian $\Delta^2 u \in L(X)$ for $u \in L(X)$ by

$$\Delta u = \partial (du), \quad \Delta^2 u = \Delta (\Delta u).$$

For $y \in Y$, let $e(y) = \{x \in X; K(x, y) \neq 0\} = \{x^+(y), x^-(y)\}$. For $a \in X$, denote by $X(a)$ the set of nodes $x \in X$ such that $K(a, y)K(x, y) \neq 0$ for some $y \in Y$.

We shall study the bi-Laplacian and bi-flows on a network by using an arc-arc incidence function $b$ on $Y \times Y$.

3. **An arc-arc incidence function**

An arc-arc incidence function $b$ on $Y \times Y$ is defined by

$$b(y, y') = \sum_{z \in X} K(z, y)K(z, y') = \sum_{z \in e(y) \cap e(y')} K(z, y)K(z, y').$$

**Proposition 3.1** The arc-arc incidence function $b$ has the following properties:

(i) $b(y, y') = b(y', y)$ for all $y, y' \in Y$;
(ii) $b(y, y) = 2$;
(iii) $b(y, y') = K(x, y)K(x, y')$ if $y$ and $y'$ meet only one node $x$, i.e., $e(y) \cap e(y') = \{x\}$;
(iv) $b(y, y') = 0$ if $e(y) \cap e(y') = \emptyset$;

In case $e(y) = e(y')$ and $y \neq y'$,
(v) \( b(y, y') = 2 \) if \( x^+(y) = x^+(y') \) and \( x^-(y) = x^-(y') \);
(vi) \( b(y, y') = -2 \) if \( x^+(y) = x^-(y') \) and \( x^-(y) = x^+(y') \).

Define a linear operator \( B_r \) from \( L(Y) \) to \( L(Y) \) by
\[
B_r w(y) = r(y) \sum_{y' \in Y} b(y, y') w(y').
\]

Lemma 3.1 \( B_r w = -d\partial w \) on \( Y \).

Proof. A simple calculation shows that
\[
B_r w(y) = r(y) \sum_{y' \in Y} \left( \sum_{z \in X} K(z, y) K(z, y') \right) w(y')
= r(y) \sum_{z \in X} K(z, y) \left( \sum_{y' \in Y} K(z, y') w(y') \right)
= r(y) \sum_{z \in X} K(z, y) \partial w(z) = -d\partial w(y). \quad \Box
\]

Define \( c(x, z) \) for \( x, z \in X \) by
\[
c(x, z) = \sum_{y \in Y} r(y)^{-1} K(x, y) K(z, y).
\]

Lemma 3.2 \( \quad \)(i) \( c(x, z) \neq 0 \) if and only if \( z \in X(x) \).

(ii) \( \sum_{z \in X} c(x, z) = 0 \).

(iii) \( \Delta u(x) = -\sum_{z \in X} c(x, z) u(z) \).

Proof. (i) It is trivial that \( z \notin X(x) \) implies \( c(x, z) = 0 \). If \( x = z \), then \( K(x, y) K(z, y) \in \{0, 1\} \) for all \( y \in Y \) and \( K(x, y) K(z, y) = 1 \) for some \( y \in Y \). Therefore \( c(x, z) > 0 \). Let \( z \in X(x) \setminus \{x\} \). Then \( K(x, y) K(z, y) \in \{0, -1\} \) for all \( y \in Y \) and \( K(x, y) K(z, y) = -1 \) for some \( y \in Y \). Therefore \( c(x, z) < 0 \).

(ii) Since \( \sum_{z \in X} K(z, y) = 0 \) for every \( y \in Y \), we have
\[
\sum_{z \in X} c(x, z) = \sum_{y \in Y} r(y)^{-1} K(x, y) \sum_{z \in X} K(z, y) = 0.
\]
(iii) \[ \sum_{z \in X} c(x, z)u(z) = \sum_{z \in X} \sum_{y \in Y} r(y)^{-1}K(x, y)K(z, y)u(z) \]
\[ = \sum_{y \in Y} r(y)^{-1}K(x, y) \sum_{z \in X} K(z, y)u(z) \]
\[ = -\sum_{y \in Y} K(x, y)du(y) = -\partial du(x) = -\Delta u(x). \quad \square \]

4. Bi-flows

Let \( a, b \in X \). We say that \( w \in L(Y) \) is a flow from \( a \) to \( b \) of strength \( I[w] \) if the following condition is fulfilled:
\[ \partial w(x) = (\varepsilon_b(x) - \varepsilon_a(x))I[w], \]
where \( \varepsilon_a(x) = 0 \) if \( x \neq a \) and \( \varepsilon_a(a) = 1 \). Denote by \( \mathbf{F}(a, b) \) the set of all flows from \( a \) to \( b \).

**Lemma 4.1** \( B_r w(y) = r(y)^{-1}(K(b, y) - K(a, y))I[w] \) for \( w \in \mathbf{F}(a, b) \).

**Proof.** We have by Lemma 3.1
\[ B_r w(y) = -d\partial w(y) = r(y)^{-1} \sum_{z \in X} K(z, y)(\varepsilon_b(z) - \varepsilon_a(z))I[w] \]
\[ = r(y)^{-1}(K(b, y) - K(a, y))I[w]. \quad \square \]

We say that \( w \in L(Y) \) is a bi-flow from \( a \) to \( b \) of strength \( J[w] \) if \( B_r w \in \mathbf{F}(a, b) \) and \( J[w] = I[B_r w] \), i.e.,
\[ \partial B_r w(x) = (\varepsilon_b(x) - \varepsilon_a(x))J[w]. \]

Denote by \( \mathbf{BF}(a, b) \) the set of all bi-flows from \( a \) to \( b \).

Assume that \( X(a) \cap X(b) = \emptyset \). We say that \( w \in L(Y) \) is a weak bi-flow from \( a \) to \( b \) of strength \( \tilde{J}[w] \) if
\[ \partial B_r w(x) = 0 \quad \text{for all } x \in X \setminus \{X(a) \cup X(b)\}, \]
\[ \tilde{J}[w] = -\sum_{x \in X(a)} \partial B_r w(x) = \sum_{x \in X(b)} \partial B_r w(x). \]
Denote by $\text{WBF}(a,b)$ the set of all weak bi-flows from $a$ to $b$.

Denote by $\mathbf{C}$ and $\mathbf{C}_B$ the set of cycles on $N$ and the set of bicycles on $N$,

\[\mathbf{C} = \{w \in L(Y); \partial w = 0\}, \quad \mathbf{C}_B = \{w \in L(Y); \partial B_r w = 0\}.\]

Denote by $\mathbf{K}_B$ and $\mathbf{H}$ the kernel of $B_r$ and the set of all harmonic functions on $X$,

\[\mathbf{K}_B = \{w \in L(Y); B_r w = 0\}, \quad \mathbf{H} = \{u \in L(X); \Delta u = 0\}.\]

**Lemma 4.2** \(\{dh; h \in \mathbf{H}\} \subset \mathbf{C} \subset \mathbf{K}_B \subset \mathbf{C}_B\).

**Proof.** Let $h \in \mathbf{H}$. Then $\partial (dh) = \Delta h = 0$, so that $dh \in \mathbf{C}$. Let $w \in \mathbf{C}$. Then by Lemma 3.1 $B_r w = -d \partial w = 0$, so that $w \in \mathbf{K}_B$. The inclusion $\mathbf{K}_B \subset \mathbf{C}_B$ is trivial. \(\Box\)

**Proposition 4.1**

(i) $\mathbf{C} \subset \mathbf{F}(a,b)$ and $\mathbf{C}_B \subset \mathbf{BF}(a,b)$ for $a, b \in X$.

(ii) $\{w \in \mathbf{F}(a,b); I[w] = 0\} = \mathbf{C}$ and $\{w \in \mathbf{BF}(a,b); J[w] = 0\} = \mathbf{C}_B$ for $a, b \in X$.

(iii) $\mathbf{F}(a,a) = \mathbf{C}$ and $\mathbf{BF}(a,a) = \mathbf{C}_B$ for $a \in X$.

(iv) $\mathbf{F}(a_1, b_1) \cap \mathbf{F}(a_2, b_2) = \mathbf{C}$ and $\mathbf{BF}(a_1, b_1) \cap \mathbf{BF}(a_2, b_2) = \mathbf{C}_B$ for $a_1, a_2, b_1, b_2 \in X$ with $\{a_1, b_1\} \neq \{a_2, b_2\}$.

**Proof.** We shall show the assertions for $\mathbf{F}(a,b)$; the assertions for $\mathbf{BF}(a,b)$ can be similarly proved. We easily have (i) and (ii).

To prove (iii), it suffices to show that $\mathbf{F}(a,a) \subset \mathbf{C}$. Let $w \in \mathbf{F}(a,a)$. Then $\partial w = (\varepsilon_a - \varepsilon_a) I[w] = 0$, so that $w \in \mathbf{C}$.

We shall prove (iv). We need to show that $\mathbf{F}(a_1, b_1) \cap \mathbf{F}(a_2, b_2) \subset \mathbf{C}$. We may assume $a_1 \notin \{a_2, b_2\}$. Using (iii) we may also assume that $a_1 \neq b_1$ and $a_2 \neq b_2$. Let $w \in \mathbf{F}(a_1, b_1) \cap \mathbf{F}(a_2, b_2)$. Then $\partial w(a_1) = -I[w]$ from $w \in \mathbf{F}(a_1, b_1)$ and $\partial w(a_1) = 0$ from $w \in \mathbf{F}(a_2, b_2)$. We have $I[w] = 0$, so that $\partial w = 0$. \(\Box\)

**Theorem 4.1** Assume that $X(a) \cap X(b) = \emptyset$.

(i) $\mathbf{BF}(a,b) \subset \text{WBF}(a,b)$ and $J[w] = \bar{J}[w]$ for $w \in \mathbf{BF}(a,b)$.

(ii) $\mathbf{F}(a,b) \subset \text{WBF}(a,b)$ and $\bar{J}[w] = 0$ for $w \in \mathbf{F}(a,b)$.

(iii) $\mathbf{F}(a,b) \cap \mathbf{BF}(a,b) = \mathbf{C}$.
Proof. It is easy to see that (i) holds. We shall prove (ii). Let \( w \in F(a, b) \).

By Lemma 4.1

\[
\partial B_r w(x) = \sum_{y \in Y} K(x, y) r(y)^{-1}(K(b, y) - K(a, y)) I[w]
\]

\[
= (c(x, b) - c(x, a)) I[w].
\]

For \( x \in X \setminus (X(a) \cup X(b)) \) we have \( \partial B_r w(x) = 0 \) by Lemma 3.2 (i). Also Lemma 3.2 (i) and (ii) show that \( \sum_{x \in X(a)} \partial B_r w(x) = -\sum_{x \in X(a)} c(x, a) I[w] = 0 \). Similarly \( \sum_{x \in X(b)} \partial B_r w(x) = 0 \).

Next we prove (iii). Lemma 4.2 and Proposition 4.1 (i) show that \( C \subset F(a, b) \cap BF(a, b) \). We shall show the converse. Let \( w \in F(a, b) \cap BF(a, b) \). Let \( x \in X(a) \setminus \{a\} \). Then the equation (1) shows that \( 0 = \partial B_r w(x) = -c(x, a) I[w] \). Lemma 3.2 (i) implies \( I[w] = 0 \), which means \( \partial w = 0 \).

**Theorem 4.2** Suppose that \( X(a) \cup X(b) \neq (X(a) \cap X(b)) \cup \{a, b\} \). Then \( F(a, b) \cap BF(a, b) \subset C \cap K_B \).

Proof. It is clear that \( (X(a) \cap X(b)) \cup \{a, b\} \subset X(a) \cup X(b) \). By our assumption, there exists \( x_0 \in X(a) \cup X(b) \) such that \( x_0 \notin (X(a) \cap X(b)) \cup \{a, b\} \). We may assume that \( x_0 \in X(a), x_0 \notin X(b) \) and \( x_0 \neq a \). Let \( w \in F(a, b) \cap BF(a, b) \). Since \( K(x_0, y) K(b, y) = 0 \) for all \( y \in Y \), we have by Lemma 4.1

\[
0 = \partial B_r w(x_0) = \sum_{y \in Y} K(x_0, y) B_r w(y)
\]

\[
= -I[w] \sum_{y \in Y} r(y)^{-1} K(x_0, y) K(a, y) = -I[w] c(x_0, a).
\]

Lemma 3.2 (i) shows that \( c(x_0, a) \neq 0 \), and that \( I[w] = 0 \). Thus \( \partial w = 0 \) on \( X \). Lemma 4.1 shows that \( B_r w = 0 \) on \( Y \).

5. **Bi-flows to the ideal boundary**

Now we recall some definitions related to the energy \( H[w] \) of \( w \in L(Y) \) and the Dirichlet sum \( D[u] \) of \( u \in L(X) \):
\[ \langle w, w' \rangle = \sum_{y \in Y} r(y)w(y)w'(y), \]
\[ H[w] = \langle w, w \rangle = \sum_{y \in Y} r(y)w(y)^2, \]
\[ L_2(Y; r) = \{ w \in L(Y); H[w] < \infty \}, \]
\[ D[u, u'] = \langle du, du' \rangle = \sum_{y \in Y} r(y)du(y)du'(y), \]
\[ D[u] = D[u, u] = H[du] = \sum_{y \in Y} r(y)(du(y))^2, \]
\[ D(N) = \{ u \in L(X); D[u] < \infty \}. \]

**Lemma 5.1** \[ \langle du, du' \rangle = -\sum_{x \in X} u(x)\Delta u'(x) \text{ for } u \in L_0(X) \text{ and for } u' \in D(N). \]

**Proof.**
\[ \langle du, du' \rangle = \sum_{y \in Y} r(y)du(y)du'(y) = -\sum_{y \in Y} \sum_{x \in X} K(x, y)u(x)du'(y) \]
\[ = -\sum_{x \in X} u(x) \sum_{y \in Y} K(x, y)du'(y) = -\sum_{x \in X} u(x)\partial du'(x) \]
\[ = -\sum_{x \in X} u(x)\Delta u'(x). \]

It is known that \( D(N) \) (\( L_2(Y; r) \) resp.) is a Hilbert space with respect to the norm \( \|u\|_2 = (D[u] + u(x_0)^2)^{1/2} \) (\( H[w]^{1/2} \) resp.) with a fixed node \( x_0 \in X \). Denote by \( D_0(N) \) the closure of \( L_0(X) \) in the Hilbert space \( D(N) \) (see [3]).

The Green function \( g_a \in L(X) \) with pole at \( a \in X \) is defined as the unique function satisfying the conditions:
\[ g_a \in D_0(N) \quad \text{and} \quad \Delta g_a = -\varepsilon_a \text{ on } X. \]

We know that \( g_a \) exists for every \( a \) if and only if \( N \) is hyperbolic, i.e., \( D_0(N) \neq D(N) \) (see [2]). Denote by \( HD(N) \) the set of all \( u \in D(N) \) such that \( \Delta u = 0 \).
Lemma 5.2 \( D_0(N) \cap \text{HD}(N) = \{0\} \) if and only if \( N \) is hyperbolic.

Proof. If \( N \) is parabolic, then \( 1 \in D(N) = D_0(N) \), which is also harmonic. This means \( 1 \in D_0(N) \cap \text{HD}(N) \).

Conversely, we assume that \( N \) is hyperbolic. Let \( u \in D_0(N) \cap \text{HD}(N) \). Then both \( u = u + 0 \) and \( u = 0 + u \) are the Royden decompositions. The uniqueness of the Royden decomposition implies that \( u = 0 \). \( \square \)

We say that \( w \in L(Y) \) is a flow from \( a \in X \) to the ideal boundary with strength \( I[w] \) if
\[
\partial w(x) = -\varepsilon_a(x)I[w].
\]
Let \( F(a, \infty) \) be the set of all flows \( w \) from \( a \) to the ideal boundary. It is well-known that \( dg_a \) is characterized as the unique optimal solution to the following extremal problem:
\[
d^*(a, \infty) = \inf \{H[w]; w \in F(a, \infty), \ I[w] = 1\}.
\]

We say that \( w \in L(Y) \) is a bi-flow from \( a \in X \) to the ideal boundary with strength \( J[w] \) if
\[
\partial B_r w(x) = -\varepsilon_a(x)J[w].
\]
Notice that
\[
J[w] = \Delta \partial w(a).
\]
Denote by \( BF(a, \infty) \) the set of all bi-flows from \( a \) to the ideal boundary of \( N \).

Analogous to \( d^*(a, \infty) \), we consider the following extremal problem:
\[
d^*_B(a, \infty) = \inf \{H[w]; w \in BF(a, \infty), \ \partial w \in D_0(N), \ J[w] = 1\}. \quad (*)
\]

The bi-harmonic Green function \( q_a \in L(X) \) with pole at \( a \) is defined by
\[
q_a(x) = \sum_{z \in X} g_a(z)g_z(x)
\]
if the sum converges (see [1], [4]). Notice that
\[ \Delta q_a = -g_a \quad \text{and} \quad \Delta^2 q_a = \varepsilon_a \text{ on } X, \]

and that \( dq_a \) is a feasible solution to the problem (*).

We proved the following lemma in [6, Theorem 4.2]:

**Lemma 5.3** Let \( N \) be parabolic and \( u \in D(N) \). If \( \sum_{x \in X} |\Delta u(x)| < \infty \), then \( \sum_{x \in X} \Delta u(x) = 0 \).

**Corollary 5.1** If \( d^*_B(a, \infty) < \infty \), then \( N \) is hyperbolic and \( \partial w = -g_a \) for all feasible solution \( w \) to the problem (*).

**Proof.** Let \( w \) be a feasible solution to the problem (*). Then \( u = \partial w \in D_0(N) \) and \( \Delta u(x) = -\partial B_r w(x) = \varepsilon_a(x) \). By the above lemma, \( N \) must be hyperbolic and \( u = -g_a \). \( \square \)

The next theorem is an extension of [4, Theorem 3.1], which shows that \( q_a \in D(N) \) is equivalent to \( q_a \in D_0(N) \).

**Theorem 5.1** The following are equivalent:

1. \( q_a \in D(N) \);
2. \( q_a \in D_0(N) \);
3. \( d^*_B(a, \infty) < \infty \).

In this case \( dq_a \) is a unique optimal solution to the problem (*).

**Proof.** It is obvious that (ii) implies (i). Suppose that \( q_a \in D(N) \). Since \( dq_a \) is a feasible solution to the problem (*), it follows that \( d^*_B(a, \infty) < \infty \). This shows that (i) implies (iii).

We shall show that (iii) implies (ii). We assume that \( d_B^*(a, \infty) < \infty \). First we shall prove that there exists an optimal solution to the problem (*). Let \( \{w_n\} \) be a minimizing sequence of (*). Then \( (w_n + w_m)/2 \) is a feasible solution to the problem (*), so that we have

\[
d_B^*(a, \infty) \leq H[(w_n + w_m)/2] \leq H[(w_n + w_m)/2] + H[(w_n - w_m)/2] \\
= (H[w_n] + H[w_m])/2 \to d_B^*(a, \infty)
\]

as \( n, m \to \infty \). Thus \( H[w_n - w_m] \to 0 \) as \( n, m \to \infty \). There exists \( w^* \in L_2(Y; r) \) such that \( H[w_n - w^*] \to 0 \) as \( n \to \infty \). Since \( \{w_n\} \) converges pointwise to \( w^* \) and \( N \) is locally finite, we obtain \( w^* \in BF(a, \infty) \) and \( J[w^*] = 1 \). Also \( \partial w_n = -g_a \) implies that
\[ \partial w^* = \lim_{n \to \infty} \partial w_n = -g_a \in D_0(N). \]

Therefore \( w^* \) is an optimal solution to the problem (*).

To prove the uniqueness of an optimal solution to the problem (*), let \( w' \) be another optimal solution to the problem (*). Then

\[
d_B^*(a, \infty) \leq H[(w^* + w')/2] \leq H[(w^* + w')/2] + H[(w^* - w')/2]
= (H[w^*] + H[w'])/2 = d_B^*(a, \infty),
\]

so that \( H[w^* - w'] = 0 \). Hence \( w^* = w' \).

For any \( \omega \in L_0(Y) \cap C(N) \) and any \( t \in \mathbb{R} \), we see that \( w^* + t\omega \) is a feasible solution to the problem (*). Thus

\[
d_B^*(a, \infty) \leq H[w^* + t\omega] = H[w^*] + 2t \langle w^*, \omega \rangle + t^2 H[\omega],
\]

so that \( \langle w^*, \omega \rangle = 0 \). By the usual way, we see that there exists \( u^* \in L(X) \) such that \( w^* = du^* \) (see the proof of [6, Theorem 3.2] for details).

Since \( D[u^*] = H[w^*] < \infty \), it follows that \( u^* \in D(N) \). Let \( u^* = v^* + h \) be the Royden decomposition with \( v^* \in D_0(N) \) and \( h \in HD(N) \). Let \( w' = dv^* \). Then \( w' \) is a feasible solution to the problem (*), so that

\[
D[v^*] + D[h] = D[u^*] = H[w^*] \leq H[w'] = D[v^*].
\]

This means that \( D[h] = 0 \) and \( H[w^*] = H[w'] \), i.e., \( h \) is a constant function and \( w^* = w' = dv^* \).

Let \( \{N_n\} \) be an exhaustion of \( N \) and \( g_a(n) \) the Green function of \( N_n \) with pole at \( a \). We have

\[
\sum_{z \in X} g_a(z)g_{x}^{(n)}(z) = -\sum_{z \in X} (\Delta v^*(z))g_{x}^{(n)}(z) = D[v^*, g_{x}^{(n)}].
\]

Since \( \{g_{x}^{(n)}\} \) converges to \( g_x \) (see [3, Section 3]), it follows that

\[
\sum_{z \in X} g_a(z)g_x(z) \leq \liminf_{n \to \infty} \sum_{z \in X} g_a(z)g_{x}^{(n)}(z) = \lim_{n \to \infty} D[v^*, g_{x}^{(n)}] = D[v^*, g_x] \leq D[v^*]^{1/2}D[g_x]^{1/2} < \infty.
\]
In particular, we obtain $\sum_{z \in X} g_a(z)^2 < \infty$, so that $q_a \in L(X)$ by [4, Theorem 2.3].

Define $f(x)$, $f_n(x)$ and $h_n(x)$ by

$$f(x) = \sum_{z \in X} g_x(z) \Delta v^*(z) = -q_a(x) \in L(X)$$
$$f_n(x) = \sum_{z \in X} g_z^{(n)}(x) \Delta v^*(z)$$
$$h_n = v^* + f_n.$$ 

Notice that $h_n$ is harmonic on $X_n$ and

$$D[h_n, f_n] = -\sum_{x \in X} (\Delta h_n(x)) f_n(x) = 0,$$

so that $D[v^*] = D[h_n] + D[f_n]$. We see by Lebesgue’s dominated convergence theorem that $\{f_n(x)\}$ converges pointwise to $f(x)$ for all $x \in X$. Since $\{D[f_n]\}$ is bounded, we see by [5, Theorem 4.1] that $q_a = -f \in D_0(N)$.

Let $f' = q_a - v^*$. Then

$$\Delta f' = \Delta q_a - \Delta v^* = -g_a + g_a = 0,$$

so that $f' \in D_0(N) \cap HD(N)$. Lemma 5.2 shows $f' = 0$. Therefore $q_a = v^* \in D_0(N)$ and $dv^* = dq_a$. \hfill \Box

6. Another extremal problem

Analogous to $d^*(a, \infty)$ and $d_B^*(a, \infty)$, we consider the following extremum problem:

$$d_B^{**}(a, \infty) = \inf \{H[w]; w \in BF(a, \infty), J[w] = 1\}. \quad (**)$$

Clearly $d_B^{**}(a, \infty) \leq d_B^*(a, \infty)$.

**Theorem 6.1** Assume that $d_B^{**}(a, \infty) < \infty$. Then there exists a unique optimal solution $w^{**}$ to the problem (**) Also there exists $v^{**} \in D_0(N)$ such that $w^{**} = dv^{**}$. 

Proof. Let \( \{w_n\} \) be a minimizing sequence of (\( \ast \ast \)). Then \( (w_n + w_m)/2 \) is a feasible solution to the problem (\( \ast \ast \)), so that we have
\[
d_{B}^{\ast}(a, \infty) \leq H[(w_n + w_m)/2] \leq H[(w_n + w_m)/2] + H[(w_n - w_m)/2] \\
= (H[w_n] + H[w_m])/2 \rightarrow d_{B}^{\ast}(a, \infty)
\]
as \( n, m \rightarrow \infty \). Thus \( H[w_n - w_m] \rightarrow 0 \) as \( n, m \rightarrow \infty \). There exists \( w^{\ast\ast} \in L_2(Y; \mathbf{r}) \) such that \( H[w_n - w^{\ast\ast}] \rightarrow 0 \) as \( n \rightarrow \infty \). Since \( \{w_n\} \) converges pointwise to \( w^{\ast\ast} \) and \( N \) is locally finite, we obtain \( w^{\ast\ast} \in \mathbf{BF}(a, \infty) \) and \( J[w^{\ast\ast}] = 1 \). Therefore \( w^{\ast\ast} \) is an optimal solution to the problem (\( \ast \ast \)).

To prove the uniqueness let \( w' \) be another optimal solution to the problem (\( \ast \ast \)). Then
\[
d_{B}^{\ast}(a, \infty) \leq H[(w^{\ast\ast} + w')/2] \leq H[(w^{\ast\ast} + w')/2] + H[(w^{\ast\ast} - w')/2] \\
= (H[w^{\ast\ast}] + H[w'])/2 = d_{B}^{\ast}(a, \infty),
\]
so that \( H[w^{\ast\ast} - w'] = 0 \). Hence \( w^{\ast\ast} = w' \).

For any \( \omega \in L_0(Y) \cap C(N) \) and any \( t \in \mathbf{R} \), we see that \( w^{\ast\ast} + t\omega \) is a feasible solution to the problem (\( \ast \ast \)). Thus
\[
d_{B}^{\ast}(a, \infty) \leq H[w^{\ast\ast} + t\omega] = H[w^{\ast\ast}] + 2t\langle w^{\ast\ast}, \omega \rangle + t^2H[\omega],
\]
so that \( \langle w^{\ast\ast}, \omega \rangle = 0 \). By the usual way, we see that there exists \( u^{\ast\ast} \in L(X) \) such that \( w^{\ast\ast} = du^{\ast\ast} \). Since \( D[u^{\ast\ast}] = H[w^{\ast\ast}] < \infty \), \( u^{\ast\ast} \in \mathbf{D}(N) \).

If \( N \) is hyperbolic type, then we let \( u^{\ast\ast} = v^{\ast\ast} + h \) be the Royden decomposition with \( v^{\ast\ast} \in \mathbf{D}_0(N) \) and \( h \in \mathbf{HD}(N) \); otherwise let \( v^{\ast\ast} = u^{\ast\ast} \in \mathbf{D}(N) = \mathbf{D}_0(N) \). Let \( w' = dv^{\ast\ast} \). Then \( w' \) is a feasible solution to the problem (\( \ast \ast \)), so that
\[
D[v^{\ast\ast}] + D[h] = D[u^{\ast\ast}] = H[w^{\ast\ast}] \leq H[w'] = D[v^{\ast\ast}].
\]
This means that \( D[h] = 0 \) and \( H[w^{\ast\ast}] = H[w'] \), i.e., \( h \) is a constant function and \( w^{\ast\ast} = w' = dv^{\ast\ast} \). \( \Box \)

We say that a network \( N \) satisfies the condition (LD) if there exists a constant \( c \) such that \( D[\Delta u] \leq cD[u] \) for all \( u \in L_0(X) \). We say that a network \( N \) is of bounded degree if \( \sup_{x \in X} \sum_{y \in Y} |K(x, y)| < \infty \).
Next proposition provides a sufficient condition for the condition (LD).

**Proposition 6.1** Assume that \( r \equiv 1 \) and that \( N \) is of bounded degree. Then \( D[\Delta u] \leq 8\nu_0^2 D[u] \) for all \( u \in D(N) \), where \( \nu_0 = \sup_{x \in X} \sum_{y \in Y} |K(x, y)| \). Especially \( N \) satisfies the condition (LD).

**Proof.** First note that a simple calculation shows that
\[
\left( \sum_{j=1}^{n} \alpha_j \right)^2 \leq n \sum_{j=1}^{n} \alpha_j^2
\]
for \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \).

Let \( w = du \) and \( v = \Delta u \). Then
\[
dv(y) = - \sum_{y' \in Y} b(y, y') w(y') = - \sum_{y' \in Y} \sum_{x \in X} K(x, y) K(x, y') w(y').
\]
Since the number of \( y' \in Y \) with \( \sum_{x \in X} K(x, y) K(x, y') w(y') \neq 0 \) is at most \( 2\nu_0 \) for each \( y \), it follows that
\[
(dv(y))^2 = \left( \sum_{y' \in Y} \sum_{x \in X} K(x, y) K(x, y') w(y') \right)^2
\leq 2\nu_0 \sum_{y' \in Y} \left( \sum_{x \in X} K(x, y) K(x, y') w(y') \right)^2.
\]
Since the number of \( x \in X \) with \( K(x, y) K(x, y') \neq 0 \) is at most two for each \( y, y' \in Y \), we have \( (\sum_{x \in X} K(x, y) K(x, y'))^2 \leq 2 \sum_{x \in X} (K(x, y) K(x, y'))^2 \).

Using \( |K(x, y) K(x, y')|^2 = |K(x, y) K(x, y')| \) we obtain
\[
(dv(y))^2 \leq 4\nu_0 \sum_{y' \in Y} \left( \sum_{x \in X} |K(x, y) K(x, y')| \right) w(y')^2.
\]
Let \( Y(x) = \{ y \in Y; K(x, y) \neq 0 \} \) for \( x \in X \). Then \( \sum_{x \in X} \sum_{y' \in Y(x)} w(y')^2 = 2 \sum_{y \in Y} w(y)^2 \). By the above estimation, we have
\[
D[\Delta u] = H[dv] = \sum_{y \in Y} (dv(y))^2
\]
\[ \leq 4\nu_0 \sum_{y \in Y} \sum_{y' \in Y} \left( \sum_{x \in X} |K(x, y)K(x, y')| \right) w(y')^2 \]

\[ = 4\nu_0 \sum_{y' \in Y} \sum_{x \in X} \left( \sum_{y \in Y} |K(x, y)| \right) |K(x, y')| w(y')^2 \]

\[ \leq 4\nu_0^2 \sum_{y' \in Y} \sum_{x \in X} |K(x, y')| w(y')^2 \]

\[ = 4\nu_0^2 \sum_{x \in X} \sum_{y' \in Y(x)} w(y')^2 = 8\nu_0^2 \sum_{y \in Y} w(y)^2 \]

\[ = 8\nu_0^2 D[u]. \]

**Lemma 6.1** Assume that \( N \) satisfies the condition (LD). If \( u \in D_0(N) \), then \( \Delta u \in D_0(N) \).

**Proof.** Let \( \{f_n\} \) be a sequence in \( L_0(X) \) such that \( \|f_n - u\|_2 \to 0 \) as \( n \to \infty \). Then \( \|f_n - f_m\|_2 \to 0 \) as \( n, m \to \infty \) and \( \{D[f_n]\} \) is bounded. By the condition (LD) there exists a constant \( c > 0 \) such that

\[ D[\Delta f_n - \Delta f_m] \leq cD[f_n - f_m] \to 0 \quad (n, m \to \infty). \]

Thus \( \|\Delta f_n - \Delta f_m\|_2 \to 0 \) as \( n, m \to \infty \). Therefore \( \{\Delta f_n\} \) is a Cauchy sequence in \( D_0(N) \). We can find \( \varphi \in D_0(N) \) such that \( \|\Delta f_n - \varphi\|_2 \to 0 \) as \( n \to \infty \). Since \( \{f_n(x)\} \) converges pointwise to \( u(x) \), it follows that \( \{\Delta f_n(x)\} \) converges pointwise to \( \Delta u(x) \). Since \( \{\Delta f_n(x)\} \) also converges pointwise to \( \varphi(x) \) and \( \{D(\Delta f_n)\} \) is bounded, we see that \( \Delta u = \varphi \in D_0(N) \) by [5, Theorem 4.1].

\[ \Box \]

**Theorem 6.2** Assume that \( N \) satisfies the condition (LD). Then

\[ d^*_B(a, \infty) = d^*_B(a, \infty). \]

If \( d^{**}_B(a, \infty) < \infty \), then \( dq_a \) is a unique optimal solution to the problem (**).

**Proof.** Since \( d^{**}_B(a, \infty) \leq d^*_B(a, \infty) \), we shall show that \( d^{**}_B(a, \infty) \geq d^*_B(a, \infty) \). We may assume that \( d^{**}_B(a, \infty) < \infty \). Let \( w^{**} \) and \( v^{**} \) be the same as in Theorem 6.1. By Lemma 6.1, we see that \( \Delta v^{**} \in D_0(N) \). This means that \( w^{**} = dv^{**} \) is a feasible solution to the problem (**). We have

\[ d^*_B(a, \infty) \leq H[w^{**}] = d^*_B(a, \infty). \]
Assume that $d^{\ast\ast}_B(a, \infty) < \infty$. Then $N$ is hyperbolic by Corollary 5.1. Let $f' = q_a - v^\ast$. Since $q_a \in D_0(N)$ by Theorem 5.1, it follows that $f' \in D_0(N)$ and $\Delta f' = \Delta q_a - \Delta v^\ast = -g_a + g_a = 0$, so that $f' \in D_0(N) \cap HD(N)$. Hence $f' = 0$. This means that $dq_a = dv^\ast$ is a unique optimal solution to the problem (**).

\[\square\]

7. An example

We show an example of $w \in BF(a, \infty)$ for the following network:

**Example 7.1** Let $X = \{x_n; n \geq 0\}$, $Y = \{y_n; n \geq 1\}$, $e(y_n) = \{x_{n-1}, x_n\}$ for $n \geq 1$. Let $K(x_n, y_n) = 1$, $K(x_{n-1}, y_n) = -1$ for $n \geq 1$ and $K(x, y) = 0$ for any other pairs. For a strictly positive function $r$ on $Y$, $N = \{X, Y, K, r\}$ is an infinite network.

Let $r_n = r(y_n)$, $R_n = \sum_{k=n+1}^{\infty} r_k$ and $\rho_n = \sum_{k=1}^{n} r_k$. We assume that $\rho := \sum_{n=1}^{\infty} r_n < \infty$. Then it is easy to see that

\[g_{x_k}(x_n) = R_n \quad (0 \leq k \leq n), \quad g_{x_k}(x_n) = R_k \quad (k > n).\]

Let $w$ be a feasible solution to the problem (***) with $a = x_0$ and let $v = \partial w$. Let $w_n = w(y_n)$ for $n \geq 1$. Let $v_n = v(x_n)$ for $n \geq 0$. We have

\[B_r w(y_n) = \frac{1}{r(y_n)} \sum_{x \in X} K(x, y_n) \partial w(x) = \frac{1}{r_n} (v_n - v_{n-1}),\]

\[\partial B_r w(x_0) = \sum_{y \in Y} K(x_0, y) B_r w(y) = -B_r w(y_1) = -\frac{1}{r_1} (v_1 - v_0),\]

\[\partial B_r w(x_n) = \sum_{y \in Y} K(x_n, y) B_r w(y) = B_r w(y_n) - B_r w(y_{n+1})\]

\[= \frac{1}{r_n} (v_n - v_{n-1}) - \frac{1}{r_{n+1}} (v_{n+1} - v_n).\]

Since $\partial B_r w(x_0) = -1$ and $\partial B_r w(x_n) = 0$ for $n \geq 1$, it follows that $r_n^{-1}(v_n - v_{n-1}) = 1$. Thus

\[v_n = \rho_n + v_0.\]
From
\[
v_n = \sum_{y \in \mathcal{Y}} K(x_n, y) w(y) = w_n - w_{n+1} \quad (n \geq 1), \quad v_0 = -w_1,
\]
it follows that \(w_n - w_{n+1} = \rho_n + v_0\), and that
\[
w_n = -\sum_{k=1}^{n-1} \rho_k - (n-1)v_0 + w_1 = -\sum_{k=1}^{n-1} \rho_k - nv_0.
\]

Let
\[
A_n = \sum_{k=1}^{n-1} \rho_k, \quad \alpha = \sum_{n=1}^{\infty} n^2 r_n, \quad \beta = \sum_{n=1}^{\infty} nr_n A_n, \quad \gamma = \sum_{n=1}^{\infty} r_n A_n^2.
\]

Then
\[
H[w] = \sum_{n=1}^{\infty} r_n w_n^2 = \sum_{n=1}^{\infty} r_n (-A_n - nv_0)^2 = \alpha v_0^2 + 2\beta v_0 + \gamma. \quad (2)
\]

Now let \(w'\) be a feasible solution to the problem (\(*\)). In a similar way we let \(w'_n = w'(y_n)\) and \(v'_n = v'(x_n) = \partial w'(x_n)\) and obtain
\[
v'_n = \rho_n + v'_0,
\]
\[
w'_n = -\sum_{k=1}^{n-1} \rho_k - nv'_0 = -A_n - nv'_0.
\]

Since \(v' \in D_0(N)\), we have \(\lim_{n \to \infty} v'_n = 0\), or \(v'_0 = -\rho\). Therefore
\[
w'_n = -A_n + n\rho. \quad (3)
\]

Since \(\rho = R_0\) and \(\rho_k = R_0 - R_k\) for \(k \geq 1\), we have
\[
w'_n = -\sum_{k=1}^{n-1} (R_0 - R_k) + nR_0 = \sum_{k=0}^{n-1} R_k. \quad (4)
\]
Notice that this is a unique feasible solution to the problem (\(*\)). By (3)
\[ d^*_B(a, \infty) = H[w'] = \sum_{n=1}^{\infty} r_n(-A_n + n\rho)^2 = \alpha \rho^2 - 2\beta \rho + \gamma. \]

(a) Assume that all of \( \alpha, \beta, \gamma \) converge. First we note that \( \alpha \rho > \beta \).

Indeed,

\[ A_n = \sum_{k=1}^{n-1} \rho_k = \sum_{k=1}^{n-1} \sum_{j=1}^{k} r_j < n \sum_{j=1}^{n} r_j = n\rho_n, \]

and that

\[ \beta = \sum_{n=1}^{\infty} n r_n A_n < \sum_{n=1}^{\infty} n^2 r_n \rho_n < \sum_{n=1}^{\infty} n^2 r_n \rho = \alpha \rho. \]

Now (2) is minimized at \( v_0 = -\beta/\alpha \), so that

\[ d^*_{B}(a, \infty) = \gamma - \frac{\beta^2}{\alpha}. \]

It follows that

\[ d^*_B(a, \infty) - d^*_{B}(a, \infty) = \alpha \rho^2 - 2\beta \rho + \frac{\beta^2}{\alpha} = \alpha \left( \rho - \frac{\beta}{\alpha} \right)^2 > 0. \]

Theorem 6.2 implies that \( N \) does not satisfy the condition (LD).

(b) Taking \( r_n = n^{-5/3} \) for \( n \geq 1 \), since \( R_n = O(n^{-2/3}) \), by (4) we have \( w'_n = O(n^{1/3}) \), and that \( H[w'] = O(\sum_{n=1}^{\infty} n^{-5/3}(n^{1/3})^2) = \infty \). This means \( d^*_B(a, \infty) = \infty \). On the other hand the bi-harmonic Green function \( q_a \) is given by

\[ q_a(x_n) = \sum_{k=0}^{\infty} g_a(x_k)g_{x_k}(x_n) = \sum_{k=0}^{n} R_k R_n + \sum_{k=n+1}^{\infty} R_k^2 = O(n^{-1/3}). \]

Thus \( q_a \in L(X) \) does not imply \( d^*_B(a, \infty) < \infty \).
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